

d) understanding decimals.

1 2 3 4 5 6 7 8 9 10 11 12 13 14

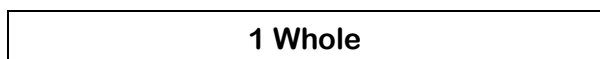
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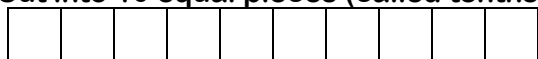
14) Recurring Decimals
13) + - x ÷ with Standard Form
12) Standard Form
11) Bounds on Calculations
10) Bounds on Rounding
9) Decimals to Fractions
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7) Rounding (to sf) & Estimation
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5) + & - Decimals
4) Decimals to %
3) x & ÷ by 10, 100 & 1,000
2) $\frac{1}{10}$, $\frac{1}{100}$, & $\frac{1}{1,000}$
1) $\frac{1}{10}$ (Tenths)
d) Decimals

Step 1) $\frac{1}{10}$ (Tenths)

A tenth is a fraction, where the whole has been broken into 10 pieces.



Cut into 10 equal pieces (called tenths)



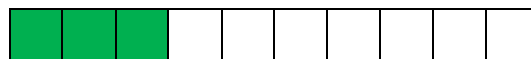
If we take 1 of these we call it one tenth

or $\frac{1}{10}$



If we take 3 of these we call it three tenths

or $\frac{3}{10}$



If we take 7 of these we call it seven tenths

or $\frac{7}{10}$



As we learnt in steps 1 & 2, with whole numbers we can write 10s, 100s and 1,000s simply by writing in the correct column,

For example 1,345 means 1 thousand, 3 hundred, 4 tens, and 5 units, which is $1000 + 300 + 40 + 5$. So with whole numbers this columns system works very well to write numbers in a short and clear way. But what about fractions. If I have 3 whole ones and $\frac{9}{10}$

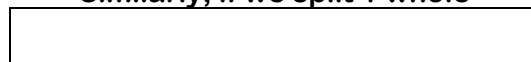
one way to do this is just to write $3\frac{9}{10}$.

IMPORTANT NOTE: in most of maths two things being next to each other means you x (times) them. There are two exceptions, digits in the decimal system (as above the value of each digit is added). And, in the case of fractions, with mixed numbers next to also means + (add).

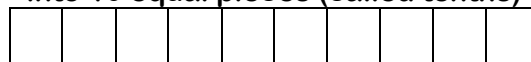
To make a neater way of writing fractions, we extend the decimal system from 1000s, 100s, 10s and 1s (units) into $\frac{1}{10}$ (tenths). We put a full stop, called a decimal point, to mark where we are moving from whole numbers into fractions.

So $3 + \frac{9}{10}$ or $3\frac{9}{10}$ can simply be written 3.9

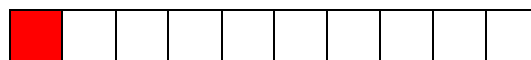
Similarly, if we split 1 whole



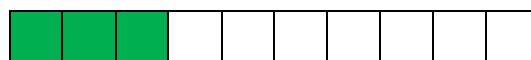
into 10 equal pieces (called tenths)



We can call $\frac{1}{10} = 0.1$



And $\frac{3}{10} = 0.3$

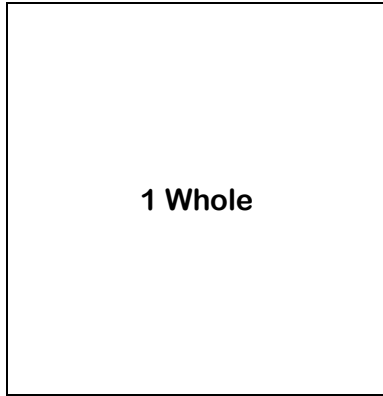


And $\frac{7}{10} = 0.7$

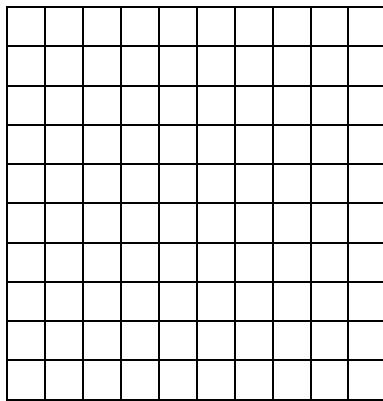


Step 2) $\frac{1}{10}$, $\frac{1}{100}$, & $\frac{1}{1,000}$

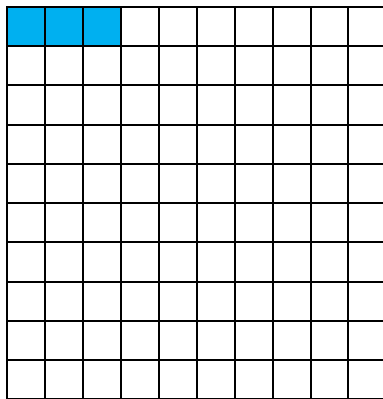
And why stop there? We don't limit our ability to talk about fractions simply to tenths. We can also work with hundredths. Let's take a slightly larger whole rectangle and cut it into 100 pieces (by cutting it into ten both ways).



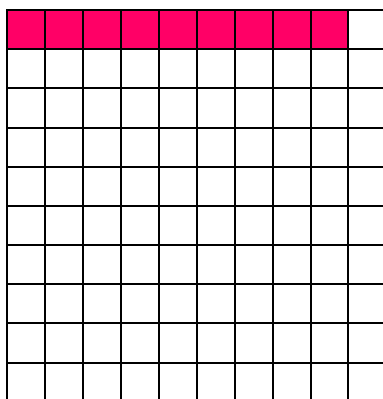
Each little square is $\frac{1}{100}$ of the large square.



Here $\frac{3}{100} = 0.03$



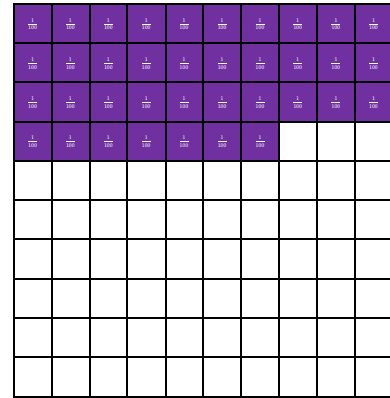
And here $\frac{9}{100} = 0.09$



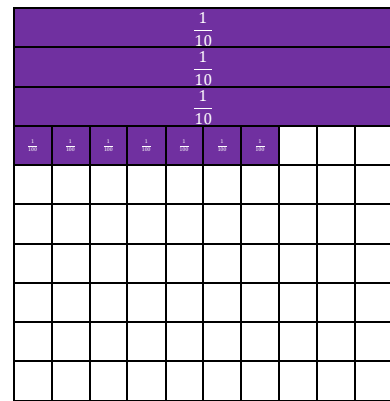
It is also worth noting that when you get 10 lots of $\frac{1}{100}$, you in fact have a $\frac{1}{10}$

$$\frac{10}{100} = \frac{10}{100} \div \frac{10}{10} = \frac{1}{10}$$

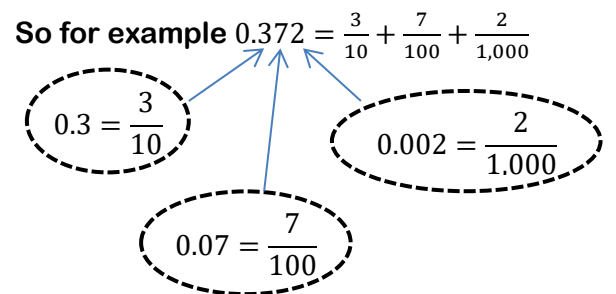
$$\frac{37}{100} = \frac{30}{100} + \frac{7}{100} = \frac{3}{10} + \frac{7}{100} = 0.37$$



=



We can go further still as the 3rd decimal column (after the decimal point) counts $\frac{1}{1,000}$ (thousandths).



Step 3) x & ÷ by 10, 100 & 1,000

With whole numbers there is a simple pattern where we can shift all the digits left to $\times 10$ and right to $\div 10$

Because $3 \times 10 = 30$
and $40 \times 10 = 400$

This means $34 \times 10 = 340$

So we can just shift digits left when multiplying. If $\times 100$ we just move the digits two spaces left (because it is the same as $\times 10$ twice).

Similarly for dividing by 10 (but moving the digits right).

$$\frac{300}{10} = 30$$

and

$$\frac{40}{10} = 4$$

so

$$\frac{340}{10} = 34$$

hence we move the digits one place to the right (or 2 places for $\times 100$ or 3 places for $\times 1,000$).

We can extend this simple way of working into the decimals.

For example.

$$0.3 \times 10 = \frac{3}{10} \times 10 = \frac{30}{10} = 3$$

and

$$0.05 \times 10 = \frac{5}{100} \times 10 = \frac{50}{100} = \frac{5}{10} = 0.5$$

We can combine these to see that $0.35 \times 10 = 3.5$

Similarly $\frac{2.6}{100} = 0.026$

In visual terms, the technique is as follows:

$$2.3 \div 10 = 0.23$$

$$\begin{array}{r} 2 \cdot 3 \\ \curvearrowright \quad \curvearrowright \\ 0 \cdot 2 \quad 3 \end{array}$$

$$0.016 \times 100 = 1.6$$

$$\begin{array}{r} 0 \cdot 0 \quad 1 \quad 6 \\ \curvearrowright \quad \curvearrowright \quad \curvearrowright \\ 1 \cdot 6 \end{array}$$

Step 4) Decimals to %

It is relatively simple to convert between decimals and %. We'll start with a %. Let's take 29%.

Now a % is a fraction over 100, and over 100 means $\div 100$, so we simply do $29 \div 100$. In step 2 we saw that this means we move the digits 2 places right, so $29\% = 0.29$

The opposite must also be true so to convert a decimal to a % we simply $\times 100$

So 0.71 as a % is
 $0.71 \times 100 = 71\%$

This pattern looks quite easy with a 2 digit %
 $23\% = 0.23$

$$87\% = 0.87$$

$$0.14 = 14\%$$

$$0.91 = 91\%$$

However we need to be careful with single digit % and other more complex decimals.

$$9\% = 9 \div 100 = 0.09$$

$$2\% = 2 \div 100 = 0.02$$

and what about a long decimal?

$$0.2461 \times 100 = 24.61\%$$

In other words, the % itself isn't necessarily a whole number, it might be a decimal!

Step 5) + & - Decimals

Just like $\times 10$ & $\div 10$, + & - decimals is closely linked to + & - whole numbers.

Let's look at $2.31 + 5.47$
 We'll break it up into parts.

Units	Tenths	Hundredths
2	$\frac{3}{10}$	$\frac{1}{100}$
5	$\frac{4}{10}$	$\frac{7}{100}$
7	$\frac{7}{10}$	$\frac{8}{100}$

$2 + 5$ $\frac{3}{10} + \frac{4}{10}$ $\frac{1}{100} + \frac{7}{100}$

So it is $7 + \frac{7}{10} + \frac{8}{100} = 7.78$
 So $2.31 + 5.47 = 7.78$

But what if you get more than 10 tenths, or more than 100 hundredths?

Let's try $5.66 + 1.97$

Units	Tenths	Hundredths
5	$\frac{6}{10}$	$\frac{6}{100}$
1	$\frac{9}{10}$	$\frac{7}{100}$
7	$\frac{6}{10}$	$\frac{3}{100}$

$\frac{6}{10} + \frac{9}{10} + \frac{1}{10}$
 $= \frac{16}{10}$
 $= \frac{10}{10} + \frac{6}{10}$
 $= 1 + \frac{6}{10}$

$\frac{6}{100} + \frac{7}{100}$
 $= \frac{13}{100}$
 $= \frac{10}{100} + \frac{3}{100}$
 $= \frac{1}{10} + \frac{3}{100}$

So $5.66 + 1.97 = 7.63$

Now everything in the hundredths column is written over 100, and in the tenths column over 10, but with the decimal system the position tells us what we are counting so we can write these out without the fractions, using just the positions to tell us what their value is.

$$\begin{array}{r} 2 \cdot 3 \quad 1 \\ 5 \cdot 4 \quad 7 + \\ \hline 7 \cdot 7 \quad 8 \end{array} \quad \& \quad \begin{array}{r} 5 \cdot 6 \quad 6 \\ 1 \cdot 9 \quad 7 + \\ \hline 7 \cdot 6 \quad 3 \\ \hline 1 \quad 1 \end{array}$$

Subtracting works in exactly the same way – and sometimes we need to borrow 10 “things” from the next column.

Borrow 1 unit = $\frac{10}{10}$

Units	Tenths	Hundredths
7 8	$\frac{17}{10}$	$\frac{9}{100}$
3	$\frac{9}{10}$	$\frac{4}{100}$
4	$\frac{8}{10}$	$\frac{5}{100}$

7 - 3 $\frac{17}{10} - \frac{9}{10}$ $\frac{9}{100} - \frac{4}{100}$

Step 6) Rounding (to dp)

If a number has lots of digits it is hard to get a sense of how big the number is. We round some of the digits, to make it easier to get a sense of its actual size. For example it is easier to sense or feel the size of 2.7 than of 2.7135

2.7 is actually 2.7135 rounded to the nearest 1 decimal place (ie the nearest $\frac{1}{10}$)

We'll look at rounding **5.8** to the nearest whole number.

4	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
5	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
6	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9

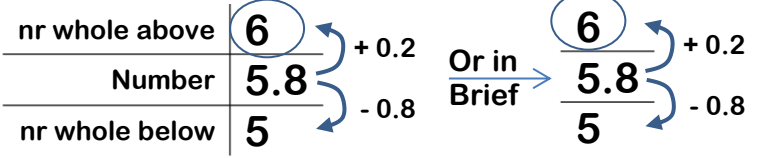
If we count up in tenths (or 0.1s) from 5.8 the nearest whole number above is **6**, we have to add 0.2 to get there.

When we count down from 5.8 the first whole number we get to is **5**. We have to count down 0.8 to get there.

And so 6 is the nearest whole number to 5.8.

Another way to write this is as follows.

Just a note that we use nr as a standard shortening for nearest when rounding.

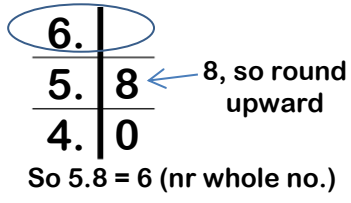


We can clearly see here again that 6 is nearer.

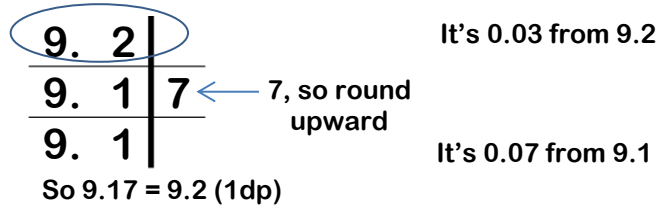
So 5.8 = 6 (nr whole number)

NOTE: It is very important that we write nr whole number in brackets! Otherwise we are saying 5.6 = 6 which is gobbledygook!

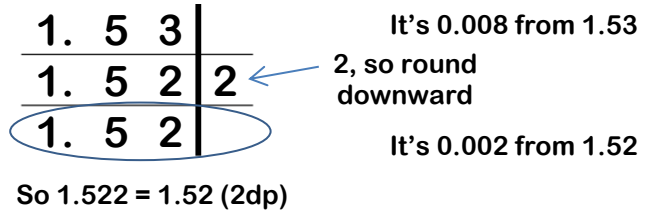
Just like with whole numbers (see +, -, x, ÷ Step 9), we only need to know the next digit to see if it rounds up or down as .5, .6, .7, .8, or .9, will always be closer to the number above, and .1, .2, .3, or .4 will always be closer to the number below.



Let's use this fastest method with a couple more examples. Round 9.17 to 1 decimal place.



Round 1.522 to 2 decimal places.



Step 7) Rounding (to sf) & Estimation

A significant figure (sf) is a digit within the number, but does not include any 0s before or after the number that are only there to tell you the position of any non-zero digits.

For example in 4,500 only the 4 and 5 are significant figures.

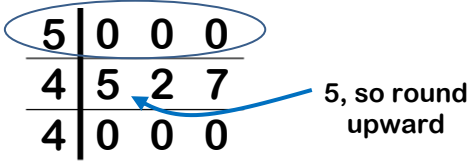
In 0.053 the 5 and the 3 are both significant figures.

In 0.0705 the 7 and the 5 are significant figures as well as the 0 in between them (because without the 0 in between the 5 would be ten times bigger!).

When we round to a number of significant figures we are rounding to the column that number of significant figures into the number.

let's round 4,527 to 1sf, 2sf and 3sf.

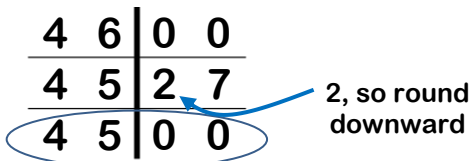
To 1sf



So 4,527 = 5,000 (1sf)
& 4,527 = 5,000 (nr 1,000)

Note that it is the same as rounding to nr 1,000.

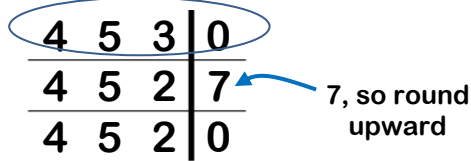
To 2sf



So 4,527 = 4,500 (1sf)
& 4,527 = 400 (nr 100)

Note it is the same as rounding it to nr 100

To 2sf



So 4,527 = 4,530 (1sf)
& 4,527 = 4,530 (nr 10)

Note it is the same as rounding it to nr 100

This works in a similar way with decimals.

So round 2.839 to 1sf is the same as rounding it to nr integer.

Rounding 2.839 to 2sf is the same as rounding it to 1pd.

Rounding 2.839 to 3sf is the same as rounding it to 2dp.

Estimations can be done quickly by rounding all the numbers in your calculation to 1sf.

If we wanted to estimate the size of a rectangular hall that is 47.87m wide and 82.39m long, we can make a reasonable estimation by doing

$$50 \times 80 = 4,000\text{m}^2$$

The exact answer is found by...
 $47.87 \times 82.39 = 3944.0093$

The estimate can be calculated mentally to...

- Get a rough idea of the size
- To check any written or calculator answers
 - For speed where an exact answer isn't necessary

Here is one more example.

Estimate the size of the answer to this calculation

$$\frac{457.8 + 723.934}{2.031}$$

We can find a quick estimate by doing

$$\frac{500 + 700}{2} = 600$$

Then if two of our friends calculated it exactly and one got 581.8 (4sf) and the other got 163,200 (4sf) we could confidently guess that 581.8 (4sf) was the correct answer as the other one is miles away from our estimate (we'd have guessed right). The second person accidentally types x into their calculator instead of +.

Step 8) x Decimals

We know that $3 \times 4 = 12$

BUT... $0.3 \times 0.4 \neq 1.2$

In fact $0.3 \times 0.4 = 0.12$

But why?

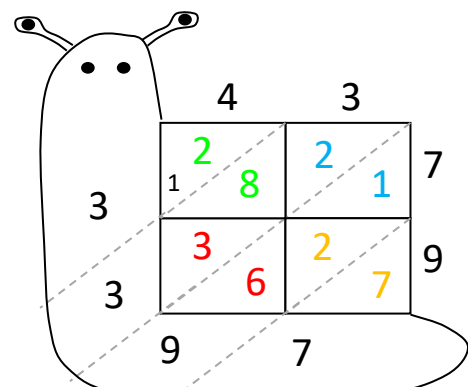
$$\text{Well } 0.3 \times 0.4 = \frac{3}{10} \times \frac{4}{10} = \frac{12}{100} = 0.12$$

This can be done for mentally single digit decimals if you can remember all the parts, but what about bigger numbers?

Let's look at 4.3×0.79

$$\begin{aligned} &= \frac{43}{10} \times \frac{79}{100} \\ &= \frac{43 \times 79}{1,000} \end{aligned}$$

We can use any method for 43×79 (here the snail).



Standard form is a way of writing very large, or very small numbers so it is easier to work with them. We write them as a number between 1 and 10, multiplied by a power of 10.

For example 5 million is 5 multiplied by 10, 6 times so it is written in standard form as follows.

$$\begin{aligned} &5,000,000 \\ &=5 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &=5 \times 10^6 \end{aligned}$$

Similarly we can work in the opposite direction, starting with a number in standard form and converting it to a decimal number.

$$\begin{aligned} &3 \times 10^4 \\ &=3 \times 10 \times 10 \times 10 \times 10 \\ &=30,000 \end{aligned}$$

Very small numbers are written as a number between 1 and 10 divided by 10 several time, in other words a negative power of 10.

$$\begin{aligned} &0.00006 \\ &=6 \div 10 \div 10 \div 10 \div 10 \div 10 \\ &=6 \times 10^{-5} \end{aligned}$$

And converting back to a decimal from standard form.

$$\begin{aligned} &7 \times 10^{-3} \\ &=7 \div 10 \div 10 \div 10 \\ &=0.007 \end{aligned}$$

We always need a number between 1 and 10. Let's work with 34,500

$$\begin{aligned} &34,500 \\ &= 3.45 \times 10 \times 10 \times 10 \times 10 \\ &=3.45 \times 10^4 \end{aligned}$$

Also $34,500 = 34.5 \times 10^3$, but 34.5 is bigger than 10

& also $34,500 = 0.345 \times 10^5$, but 0.345 is less than 1.

$34,500 = 0.345 \times 10^5 = 3.45 \times 10^4 = 34.5 \times 10^3$
but only 3.45×10^4 is in standard form as 3.45 is the only number between 1 and 10.

In fact it wouldn't be a "standard" form if there were different ways to write it!

Similarly, with 0.0058 we'd use 5.8, as this uses the same digits but is between 1 and 10 so...

$$\begin{aligned} &0.0058 \\ &=5.8 \div 10 \div 10 \div 10 \\ &=5.8 \times 10^{-3} \end{aligned}$$

Step 13) + - x ÷ with Standard Form

Let's look at adding numbers that are written in standard form.

$$3 \times 5 + 4 \times 5 = 7 \times 5$$

When we add multiples of five, we just get a bigger multiple of 5.

We can do something similar with multiples of a power of 10.

$$3 \times 10^3 + 4 \times 10^3 = 7 \times 10^3$$

This makes sense because $3,000 + 4,000 = 7,000$

This will work just as well with more digits.

$$1.2 \times 10^3 + 2.4 \times 10^3 = (1.2 + 2.4) \times 10^3 = 3.6 \times 10^3$$

Let's check this....

$$1,200 + 2,400 = 3,600 \dots \text{yay, it works!}$$

We can also use this to add large numbers written in standard form of similar, but not the same order of magnitude.

$$3.4 \times 10^7 + 2.7 \times 10^6$$

|They're not quite in the same power of 10, but we can convert them so they are. Let's put them both as a multiple of 10^7 .

$$2.7 \times 10^6 = 0.27 \times 10^7$$

This is no longer in standard form (as 0.27 is less 1) .but it will allow us to add the two numbers

so

$$\begin{aligned} &3.4 \times 10^7 + 2.7 \times 10^6 \\ &=3.4 \times 10^7 + 0.27 \times 10^7 \\ &=(3.4+0.27) \times 10^7 \\ &=3.67 \times 10^7 \end{aligned}$$

Be careful that your final answer is in standard form. If we had done it by converting the power in 3.4×10^7

$$3.4 \times 10^7 = 34 \times 10^6$$

so

$$\begin{aligned} &3.4 \times 10^7 + 2.7 \times 10^6 \\ &=34 \times 10^6 + 2.7 \times 10^6 \\ &=36.7 \times 10^6 \end{aligned}$$

This is a correct addition but the final answer isn't in standard form. We need to convert it..

$$\begin{aligned} &3.4 \times 10^7 + 2.7 \times 10^6 \\ &=34 \times 10^6 + 2.7 \times 10^6 \\ &=36.7 \times 10^6 \\ &=3.67 \times 10^7 \end{aligned}$$

With subtracting we can work in the same way.

$$\begin{aligned} &5.9 \times 10^9 - 2.4 \times 10^8 \\ &=5.9 \times 10^9 - 0.24 \times 10^9 \\ &=5.66 \times 10^9 \end{aligned}$$

For multiplying numbers in standard form we just rearrange the order of the parts.

$$\begin{aligned} &(2.9 \times 10^5) \times (5.3 \times 10^3) \\ &2.9 \times 10^5 \times 5.3 \times 10^3 \\ &=2.9 \times 5.3 \times 10^5 \times 10^3 \\ &=(2.9 \times 5.3) \times (10^5 \times 10^3) \\ &=15.37 \times 10^8 \end{aligned}$$

Which in standard form
 $=1.537 \times 10^9$

And with division similarly

$$\frac{2.7 \times 10^8}{3 \times 10^5}$$

$$= \frac{2.7}{3} \times \frac{10^8}{10^5}$$

$$= 0.9 \times 10^3$$

Which in standard form is

$$= 9 \times 10^2$$

It is tempting here as this is such a small number to write 900 instead of 9×10^2 , and although they have the same value and meaning, they are not in the same form, so when working with standard form we would leave the answer as 9×10^2 .

Step 14) Recurring Decimals

We already know that

$$0.333333 \dots = 0.\dot{3} = \frac{1}{3}$$

&

$$0.666666 \dots = 0.\dot{6} = \frac{2}{3}$$

Here we will look at recurring decimals in more detail and understand why they recur using algebra.

Let's look at why $\frac{1}{9} = 0.111111 \dots = 0.\dot{1}$

On a simple level we can use the bus stop method.

$$\begin{array}{r} 0 \cdot 1 \quad 1 \quad 1 \quad 1 \dots \\ 9 \overline{) 1 \cdot 0 \quad 0 \quad 0 \quad 0} \end{array}$$

This can be used for any recurring decimal proof

such as $\frac{5}{9} = 0.555555 \dots = 0.\dot{5}$

$$\begin{array}{r} 0 \cdot 5 \quad 5 \quad 5 \quad 5 \dots \\ 9 \overline{) 5 \cdot 0 \quad 0 \quad 0 \quad 0} \end{array}$$

but will be much harder work for 99^{ths} or 999^{ths} for which we can (and should!) use a much cleverer but

simpler algebraic method. Let's look at $\frac{7}{9} =$

$0.77777 \dots = 0.\dot{7}$ to explain how this works.

Do	Equation	Eq ⁿ Name
Call decimal x	$x = 0.7777 \dots$	a
10a	$10x = 7.7777 \dots$	b
b - a	$9x = 7$	c
c ÷ 9	$x = \frac{7}{9}$	

This wonderful method works for any 9^{ths} , but also for decimals that repeat every 2nd digit (99^{ths}) every 3rd digit (999^{ths}) and so on.

Let's use this algebraic method to show that

$$\frac{43}{99} = 0.434343 \dots = 0.\dot{4}\dot{3}$$

Do	Equation	Eq ⁿ Name
Call decimal x	$x = 0.4343 \dots$	a
100a	$100x = 43.4343 \dots$	b
b - a	$99x = 43$	c
c ÷ 99	$x = \frac{43}{99}$	

This same structure can even be used for 999^{ths} and beyond.

$$\frac{771}{999} = 0.771771 \dots = 0.\dot{7}\dot{7}\dot{1}$$

Do	Equation	Eq ⁿ Name
Call decimal x	$x = 0.771771 \dots$	a
1000a	$1000x = 771.771771 \dots$	b
b - a	$999x = 771$	c
c ÷ 999	$x = \frac{771}{999}$	

Abracadabra – it's magic!