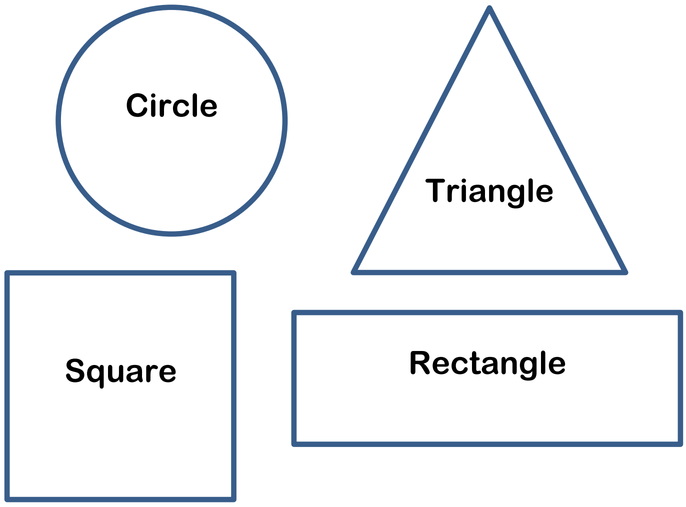
k) Angles

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

15) Sin, Cos & Area Rules
14) Trig Graphs
13) Circle Theorems
12) 3D Pythag & Trig
11) Trigonometry
10) Pythagoras
9) Parallel Line Rules (X, F & Z)
8) Angle Sum Rules
7) 3D Shapes
6) Polygon Types
5) Quadrilateral Types
4) Triangle Types
3) Angle Types
2) Angles as Turn
1) Basic Shapes (Sketch)
k) Angles

1) Basic Shapes

At this step we are able to see the difference between the three core basic shapes... the circle, the triangle, the square, and the rectangle.



The circle is the round shape, like bike wheels, hula hoops, and buttons on clothes.

The triangle is the shape with three sides. Tri mean three – like a three legged tripod that a camera stands on to take photos without a wobbling photographer.

The square is a shape with four equal sides, and a rectangle also has four sides, and is a bit like a longer, thinner version of a square.

2) Angles as Turn

This is the absolute key step in the whole of the angles topic – and sadly, the one that is missing in almost all forms of mathematical education. In my 15 years of teaching maths in schools and outside of them, I never once met a student or adult who could tell me what an angle actually is.

Angles are generally understood as numbers in the corner of shapes, near the vertex. A few people get as far as answering the question by saying the corner of a shape measured in degrees.

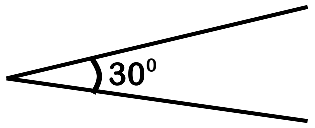
BUT WHAT DOES AN
ANGLE ACTUALLY MEASURE???

The answer very simply is turn or rotation.

If you turn all the way round to again face the same way you started you have turned through 360^0 (read 360 degrees). This is called a full **TURN.** Despite being able to quote that 360^0 is a full turn, and half of that, 180^0 is a half turn and so on – every learner I have met, by the end of GCSE or A-level maths, still hasn't understand angles as a measure of turn.

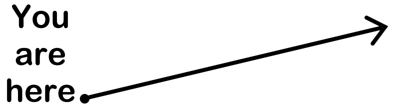
So an angle of 360^0 is a full turn from wherever you were facing at the start (and you end up facing that same way). All other angles are parts or multiples of this. 1^0 is $\frac{1}{360}$ of a full turn. It is a very small amount of turn, but if you turn that small amount 360 times you have actually turned through a full turn.

These are represented as lines with a little curve. Let's look at how 30^0 is represented.

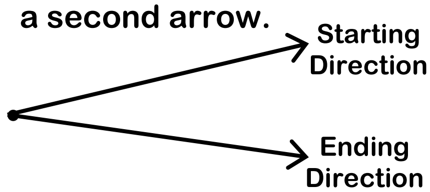


And you can see why, without any further explanation, folks think of angles as measurement of degree in the corner of a shape. But the missing information here is this.

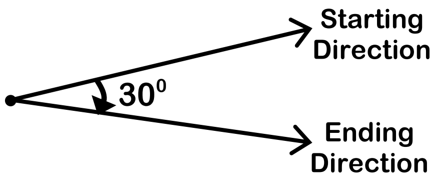
Start at a point facing in a particular direction, and mark the direction you are facing with an arrow...



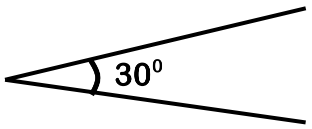
Then turn 1^0, then turn another degree. Keep turning until you have turned through 30^0, and then mark the new direction you are facing with a second arrow.



And then finally, to mark the angle of 30^0 draw a little curved arrow near the vertex on which you are standing, and label it by writing 30^0.



Now if all angles were labelled as above, then it would be easy when working with angles to remember that they represent a turn. Starting from facing along the top arrow, we turn around the curved arrow through 30^0 until we end facing the direction of the second arrow. However, to save time, we don't write the dot where you are standing, or the arrow heads that show the different directions you face, or the way you turn. Aaaaah!!!



So please remember, an angle is a measure of turn. It is represented by a little curved line (often with the number of degrees you have turned through labelled next to the little curved line). The two lines the curve goes between are the direction you face before and after the turn. The point you stand when turning is represented by the vertex where the lines touch.

You have to imagine the three arrow heads on each of these lines!

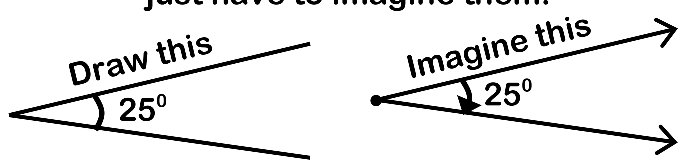
Finally, it doesn't matter if we turn clockwise or anti-clockwise. If we start at the top line and turn clockwise 30^0 we end up facing along the bottom line. If we start facing along the bottom line and turn 30^0 anti-clockwise we end up facing along the top line. Angles don't have a direction, they simply say that to turn from facing along one line, to the second line (standing at the vertex where the lines meet) you would need to turn through a certain number of degrees.

3) Angle Types.

There are several types of angles, which are named according to their size (ie how far you turn).

The smallest type of angles are called acute angles, cause they are really cute. They are less than a quarter of a full turn (less than 90^0).

Here is an example of 25^0. Remember that it would be easier to understand this as a turn of 25^0 if the arrowheads were drawn on, but you just have to imagine them.

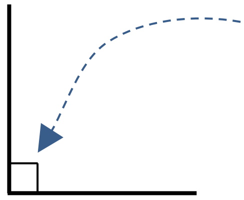


Note that things that are small always look cute... hence acute angles are $<90^0$.

Cute

Other examples of acute angles are 1^0, 33^0, 58^0, 72^0, & 99^0.

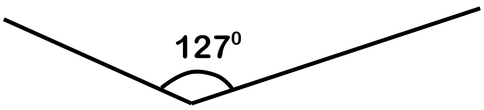
The next type of angle is called a right angle. A right angle I always bigger than any acute angle (it has more turn). A quarter turn, or 90^0, is considered to be such a special angle, that it has its own special name. We call it a right angle. It also has its own special symbol. Instead of the curvy line (which ideally would have an arrow head but doesn't) that we use for *all other angles*, for a *right angle*, we use two straight lines that are at right angles. (To be "at right angles," means that you have to turn 90^0 to turn from one lines direction to the others).



A right angle, or 90^0, doesn't even need a number label. Whenever you see the "square" angle label, it's a right angle.

Anything larger than a right angle (but smaller than a half turn) is called an obtuse angle. Remember that an obese person is very large, an obtuse angle is also very large (but not as large as a reflex angle as we'll see shortly).

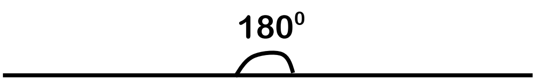
An example of an obtuse angle is 127^0.



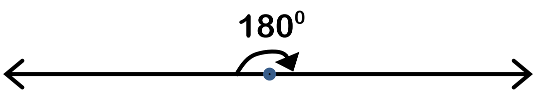
Some more examples of obtuse angles are 91^0, 97^0, 103^0, 148^0, & 179^0.

A half turn is bigger than any acute, right or obtuse angle. A half turn is always exactly 180^0.

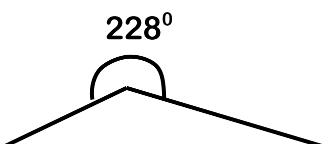
Without understanding angles as turn, 180^0 is often mistakenly described as a straight line. It is actually the turn from a point on a line from facing one way along the line, to facing the opposite way along the line.



You can see why it gets called a straight line (rather than angles at a point on a straight line). The point & arrows, as always, make this clearer.



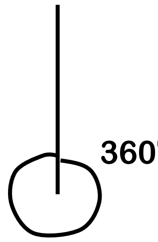
Reflex angles are larger than any acute or obtuse angles, or a half turn. These are larger than 180^0. An example is 228^0.



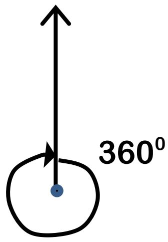
Other examples of reflex angle are 181^0, 217^0, 284^0, 299^0, 312^0, & 359^0.

Finally a full turn is bigger than all of these others, it is exactly 360^0. This is often mistakenly called a circle, but it is a full turn, where you end up facing the same way you started. The curved line that represents the angle, looks like a circle, which is where this error comes from.

A full turn



Again, points and arrows make it clearer that this is a full turn, not a circle!

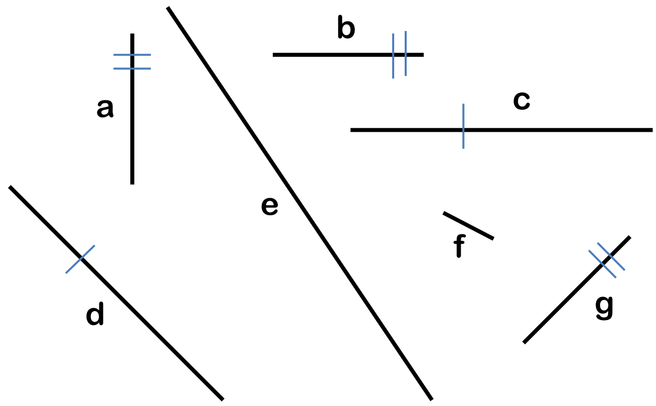


4) Triangle Types

There are two main types of triangles, those where all the sides and angles are different (called scalene triangles), and those that have some sides and angles that are the same (called isosceles triangles).

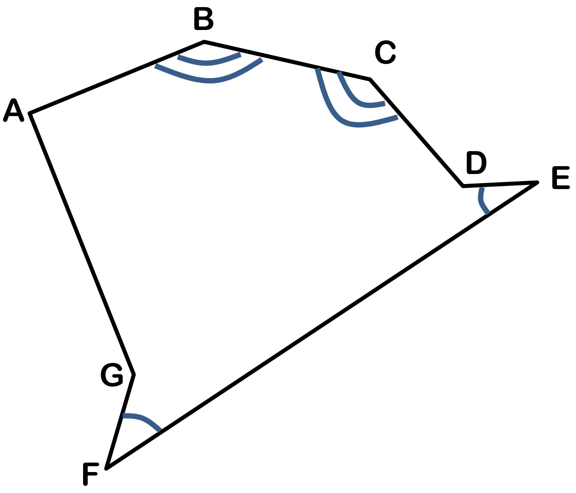
Firstly it is useful to have a shorthand way of labelling same sides and same angles (when we say sides, we always mean the same length sides).

To label sides of the same length we put a small dash across each line. If there is a second group of lines of the same length we can put two dashes across each line of that length.



Lines a, b & g are the same length as they each have two dashes across them. Similarly lines c & d are the same as they both have one dash across them.

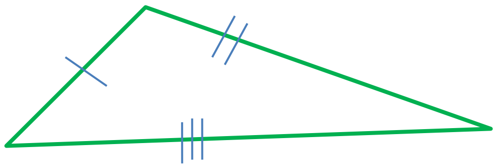
We label equal angles with the same number of angle curves.



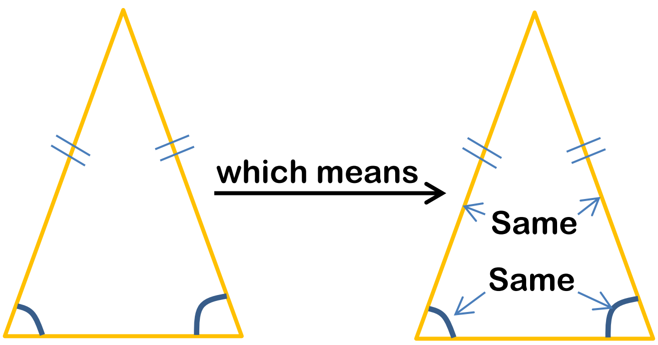
So here the angles at vertex E and vertex F are the same as they have one angle curve. The angles at vertex B & vertex C are also the same as they have two angle curves.

So with this we can now sketch the types of triangle.

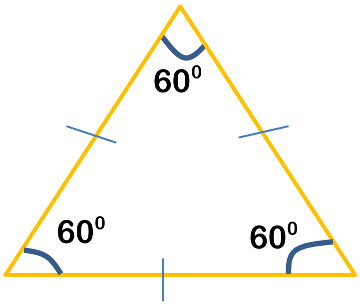
A scalene triangle is one that where all three sides and angles are different.



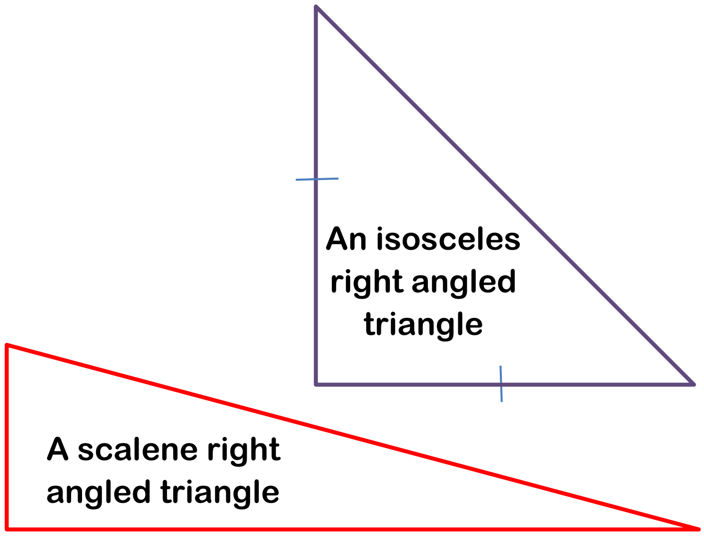
The other type of triangle does have sides and angles that are the same – this is called an isosceles triangle. The two sides that are the same, will also correspond to a pair of angles that are the same!



There is a very special type of isosceles triangle, called an equilateral triangle, where the third side and angle, are the same as the other two. In this case all three angles are always 60^0 (see step 8 for why). Another name for it is a regular triangle.



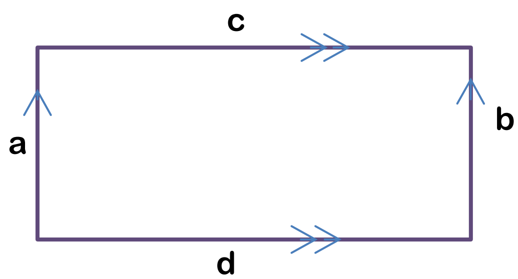
There is one other type of triangle, called a right angled triangle. This simply means that one of the right angles is a right angle. You can't have two right angles in the same triangle because the ends wouldn't meet!



5) Quadrilateral Types

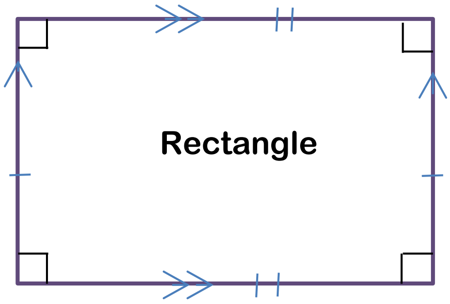
Just like a quad bike is a type of motor bike with 4 wheels, and quadruple means to multiply by 4, a quadrilateral is a shape with four (straight) sides.

There are several types, and some of their properties are to do with parallel sides. Two parallel lines can be extended forever without crossing, like train tracks that never meet. The distance between two parallel lines will always be the same. They are represented on a sketch with the same number of arrows.

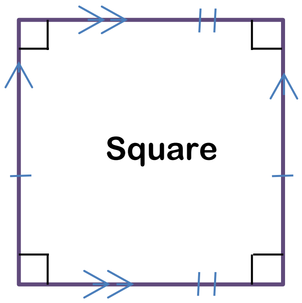


So sides a & b are parallel, and c & d are parallel.

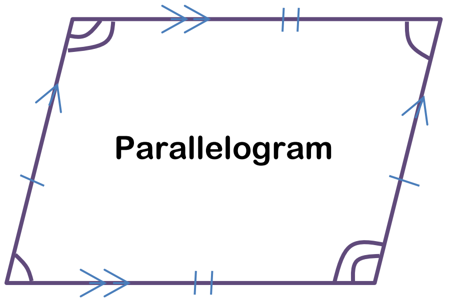
The two you already know from step one are squares and rectangles. A rectangle has four right angles and each pair of opposite sides is both parallel and of equal length. A square is a special type of rectangle with all four sides being equal.



A square is a regular quadrilateral (all sides and angles are equal).

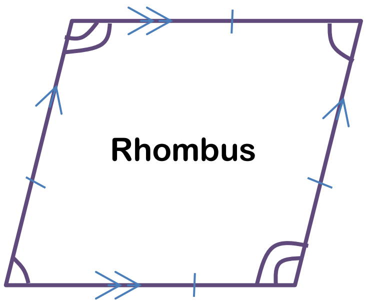


When you tip a rectangle over, keeping the parallel opposite sides and the equal opposite size but changing the right angles you get a parallelogram. In a parallelogram the opposite sides are equal.



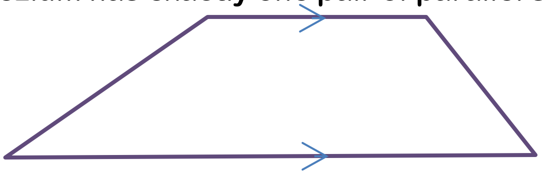
One way to remember this is the phrase rectanglellogram.

If you do the same thing with a square, you create a rhombus. This similarly has no right angles, but opposite sides are parallel, opposite angles are equal and all four sides are equal.

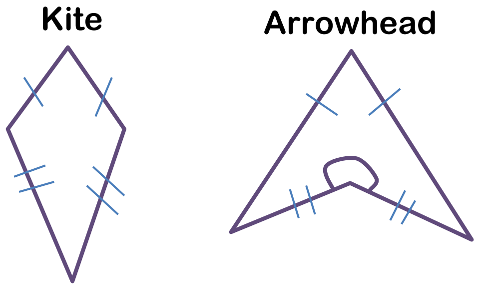


Rhombusqaure and squarhombus can with remembering this (they start and end with each-others first and last sounds).

A trapezium has exactly one pair of parallel sides.



Lastly with have a kite and an arrowhead. These don't have any pairs of parallel sides, but each have two pairs of adjacent equal sides. The kite has no reflex angles, the arrowhead has one.

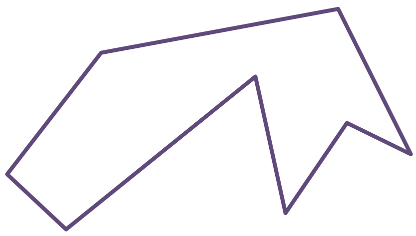


6) Polygon Types

A polygon is a shape with straight sides. We have already met triangles (3-sided polygons) and quadrilaterals (4-sided polygons). They are named by the number of sides and internal angles they have.

No Sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

What type of shape is this?

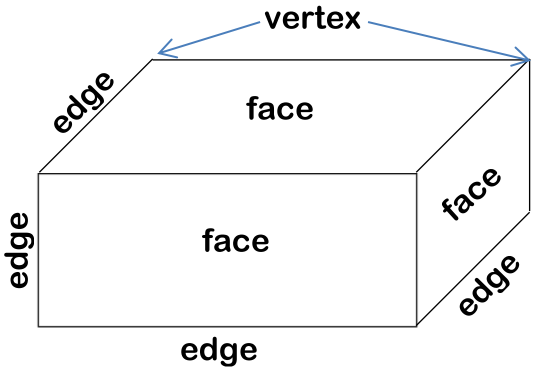


Counting up the sides (or angles) there are 8 so it is an octagon.

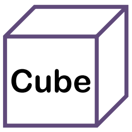
Please note that because we so often see pictures of regular polygons, where all sides and angles are the same, folks sometimes get confused and think all polygons are regular, but polygons can be super zig-zaggy (see the above octagon) as long as all the sides are straight!

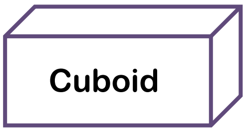
7) 3D Shapes

3D shapes are made up of vertices (a vertex is a point where two edges meet), edges (where two faces meet) and faces. The faces are often polygons but could also be, for a example a circle.



A cube is a 3D shape in which every face is a square. A cuboid is a 3D shape in which every face is a rectangle.

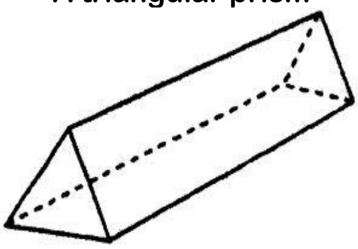




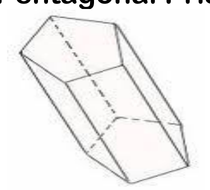
Prisms are 3D shapes, such that wherever you cut them (called the cross-section) the shape you cut across is the same. A cuboid could also be called a rectangular prism as wherever you cut it you get a rectangle. A cube could be called a square prism as wherever you cut it you get a square.

Similarly we have....

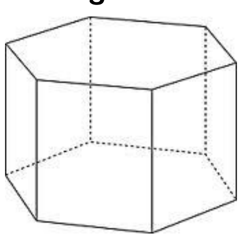
A triangular prism



A Pentagonal Prism



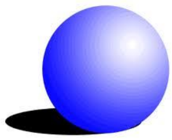
A Hexagonal Prism



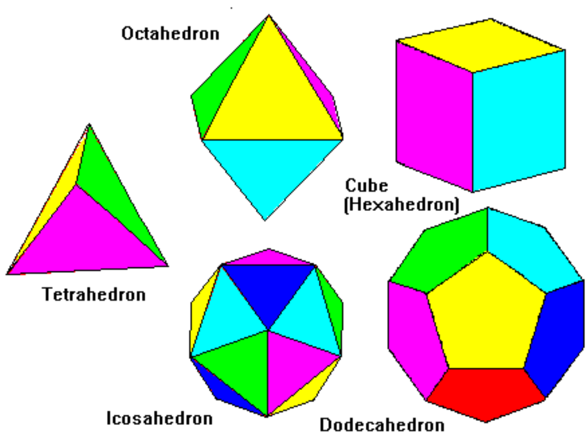
Or even a bread shaped prism!

There is also a special type of round prism called a cylinder (or circular prism) – wherever you cut it you get a circle.

A fully 3D circle (informally called a ball or globe) is called a sphere...



For the very keen 3Der it is useful to know the platonic solids. They were discovered by someone called Plato (hence the name). They are 3D shapes where all the faces are regular polygons.

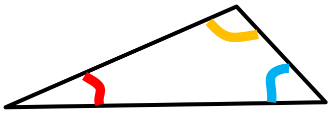


Plato believed that you can't make ANY OTHER 3D shapes out of regular polygons – can you find one that disproves his belief? Here is some more info about Plato's wonderful Platonic solids.

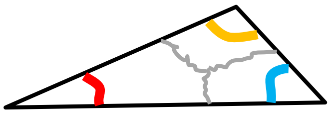
Name	No of Faces	Shape of Faces
Tetrahedron	4	Regular (Equilateral) Triangle
Cube (Hexahedron)	6	Regular Quadrilateral (Square)
Octahedron	8	Regular (Equilateral) Triangle
Dodecahedron	12	Regular Pentagon
Icosahedron	20	Regular (Equilateral) Triangle

8) Angle Sum Rules

The angle sum rules are brilliant! It turns out that in a triangle the 3 angles ALWAYS add up to 180^0. The simplest way to understand this is by drawing a triangle and then ripping off the angles and putting them together.

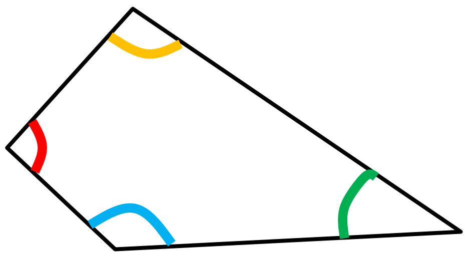


Firstly rip off the three angles.

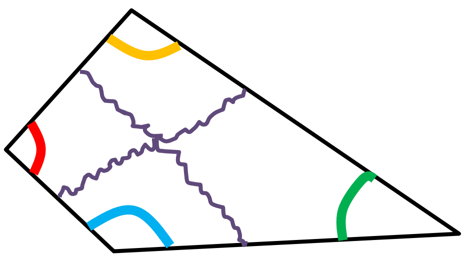


And then put the 3 angles together at a point and… Abracadabra, in total they make a half turn because they sit on a line..

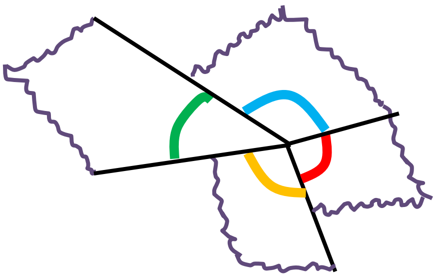
Now let's look at a quadrilateral (a 4 sided polygon). Incredibly the angles in a 4-sided shape always add up to 360^0. This can be understood in the same way as for a triangle.



Similarly tear the fear angles apart.



And then put the 4 angles around a point.



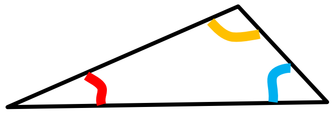
You will find that they make a full turn, so the four angles add up to 360^0

Now this is all very exciting but it doesn't prove it for sure. You can become more convinced by getting a large group of friends to do this and you will discover that for every single triangle, when you put the angles together they make a half turn or 180^0 and for every single quadrilateral when you put the angles together they make a full turn or 360^0.

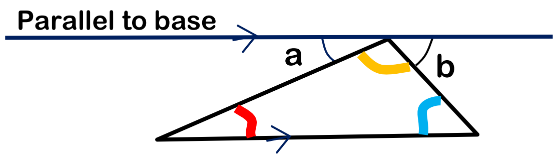
This is fantastic, amazing, fascinating, magical, and a whole host of other things… but it still isn't certain. Even if we get 100 friends to do this and they all get the same answer – we become much more sure, but not certain!

Let's cut straight to the proof… it relies on step 9 as it uses the alternating angles "Z" rule. You might need to spend some time on this first or come back to this later.

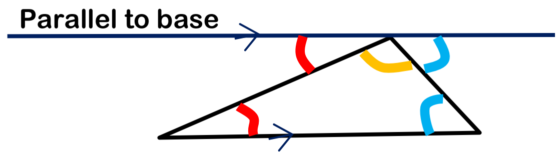
Let's use our triangle from earlier in the step.



Draw a line parallel to the "bottom" line, just touching the "top" vertex. This gives us two new angles a & b.

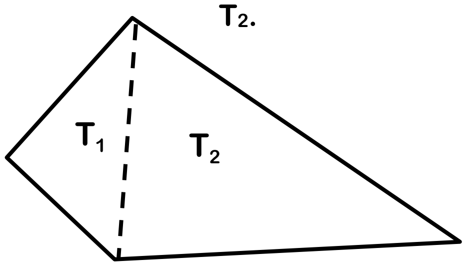


And because of the two parallel lines, we know that a is the same as the red angle (by the alternating angles "Z" rule), and b is equal to the blue angle (by the alternating angles "Z" rule).



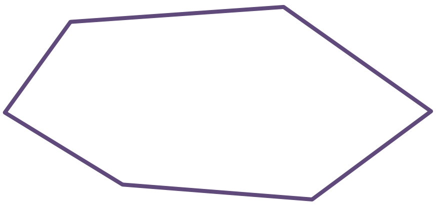
And so the 3 angles red, blue and green fit on a straight line at a point, so they make a half term so add up to 180^0. But they are the same 3 angles red, blue and green that are in the triangle so they must also add up to 180^0. You can do this for any triangle so the angles in any triangle add up to 180^0.

And this also proves the rule for a quadrilateral. For any quadrilateral can be cut into 2 triangles T_1, and T_2.

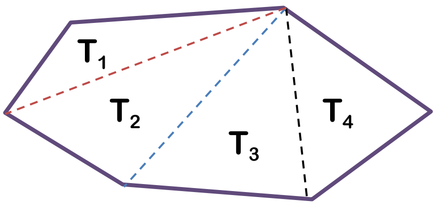


The angles in each of these triangles add up to 180^0 as above, and together they make up all the angles of the quadrilateral. There are two pairs of adjacent angles from the neighbouring triangles that make up one of the quadrilateral angles. Hence the 6 triangles angles, sum to make the 4 quadrilaterals angles. So the angles in T_1 plus the angles in T_2 make the angles in the whole quadrilateral. And $180^0 + 180^0 = 360^0$ So we now know that the angles in a quadrilateral will always add up to 360^0 (because they are the sum of the 6 angles that make up two triangles).

You can use this same method for any polygon, breaking it into a number of triangles. Let's look at this hexagon as an example.



We can break it into 4 triangles.



These 12 angles from these 4 triangles, in total make the 6 angles from our hexagon (some pairing up to form one angle in the bigger polygon). Hence the sum of the angles in these 4 triangles, is the same of the sum of the angles in the hexagon.

So the sum of a hexagon's angles is $4 \times 180^0 = 720^0$

Let's do this for all the polygons, starting with one triangle, and building up, splitting larger polygons into triangles to find their angle sums.

Here's a table of all the results.

Polygon	Sides	Triangles	Calculation	Angle Sum
Triangle	3	1	1×180^0	180^0
Quadrilateral	4	2	2×180^0	360^0
Pentagon	5	3	3×180^0	540^0
Hexagon	6	4	4×180^0	720^0
Septagon	7	5	5×180^0	900^0
Octonagon	8	6	6×180^0	1080^0
Nonagon	9	7	7×180^0	1260^0
Decagon	10	8	8×180^0	1440^0

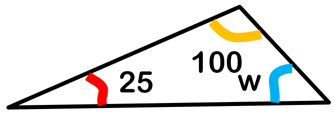
You can see that for every extra side, you add one more triangle to your polygon, and hence 180^0 to the angle sum.

In general there are always two less triangles than sides. So for an n sided shape, there are n – 2 triangles. Each of the n – 2 triangles has sum 180^0 so for an n sided shape we must do 180(n – 2).

So the angle sum of an n sided shape is…

$$180(n-2)$$

We've come a long way in one step! Let's look at an example for a triangle.

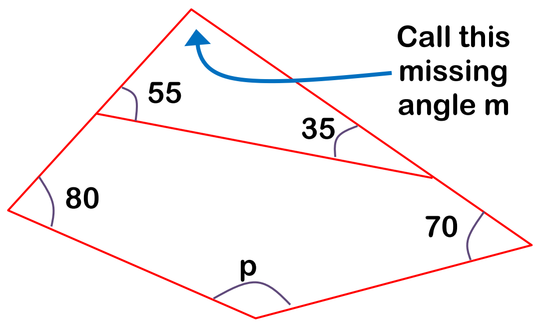


Now because it is a triangle we know that the three angles must add up to 180.

The missing number that makes this add up to 180 is 55, as 100 + 25 + 55 = 180, so w = 55. We can also solve this more formally using equations…

Equation	Action
w + 100 + 25 = 180	Sum LHS constants
w + 125 = 180	-125
w = 55	

One of the most important things with angles is the idea of a multi-step question. Most problems we want to solve don't just involve one triangle, with 2 known angles and one unknown angle. They might involve several shapes and a number of known and unknown angles. This means that we might have to find several missing angles before we can find out the one we really want to know. Sometimes in these different steps of learning, we might even have to use different steps of a ladder.



If we knew the missing angle m, then we'd have three known angles and p to make up the outer quadrilateral. Knowing that those four angles must add up to 360 we could then find p. But we don't know the top angle (that we have called m here).

However, this angle we've called m, is in a triangle, where the other two sides are known. m plus the other two angles 55 and 35 must add up to 180, as they are within a triangle. So m us 90 (a right angle!).

Now we know three of the four angles in our quadrilateral. 90, 80 and 70, as well as p. 90 + 80 + 70 is 240, so we need another 120 to make 360, hence p is 120. Or using equations…

p + m + 80 + 70 = 360
But we have two missing angles.

But working within the triangle

Equation	Action
m + 55 + 35 = 180	Sum LHS constants
m + 90 = 180	- 90
m = 90	

Then within the whole quadrilateral

Equation	Action
p + 90 + 80 + 70 = 360	Sum LHS constants
p + 240 = 360	- 240
p = 120	

Let's try and solve a problem with an 8 sided polygon. Our octagon has 6 angles of one size, and two angles of another size. Each of the 6 identical angles are 140^0, how big are the pair of identical angles?

Makes 6 triangles	Total
6 x 180^0	1080^0

So looking it up in our table we see that the angle sum for an 8 sided polygon is 1080^0. Let's just show how to use the formula (we might not have a table).

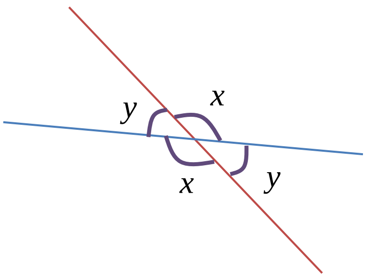
Angle Sum = 180(8 – 2)
= 180 x 6
= 1080^0

Now 6 of the angles are each 140^0, so the sum of those 6 angles is 6 x 140 = 840.

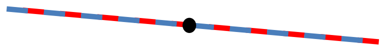
So the final 2 angles must have a sum of
1080 – 840 = 240
As they are two identical angles, each one must be half of 240, which is 120^0

9) Parallel Line Angle Rules (X, F & Z)

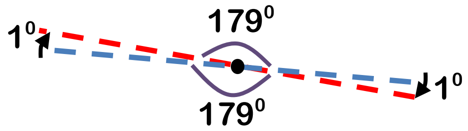
The X rule is formally called the vertically opposite angles rule, but for most people opposite angles is enough to understand what the rule is talking about. Where any two lines cross (making an "x" shape) the opposite angles are the same.



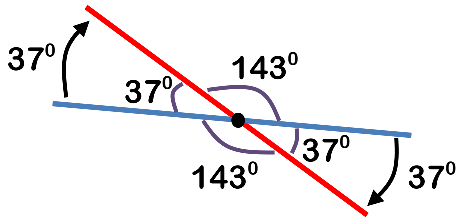
But why? Let's think of these two lines as lying on top of one another. Then the angle between them, at the point labelled with a black dot, is 0^0, 0r 180^0 depending on which way you look at it!



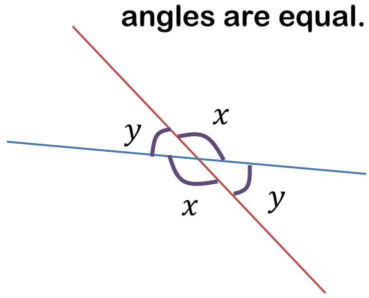
But if I turn one end of the red line 1^0 clockwise (centred around the black dot), then the other end of the red line also turns exactly 1^0 clockwise (otherwise it would no longer be a straight line). Now if the blue line stays put, then the angle between the blue and the red lines must now be 1^0 (leaving the other two angles being 179^0).



Similarly, if we turn one end of the red line (from directly over the blue line), 37^0 clockwise, then the other end will also turn by 37^0 (around the black dot, relative to the blue line).

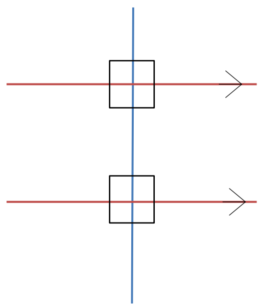


You will end up with the opposite angles both being 37^0, as well as another pair of equal opposite angles of 143^0. This works however much you turn the red line, relative to the blue line. So the opposite angles must always be equals. Hence the vertically opposite angles rule, or "X" rules says that for any pair of intersecting lines, the opposite pairs of angles are equal.

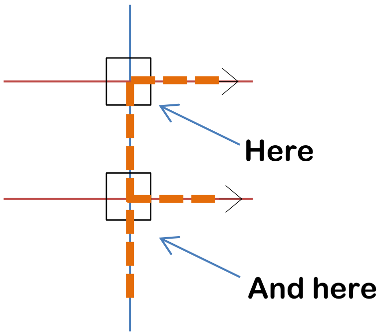


Now let's look at the corresponding angles, or as it is informally known, the F-rule.

Let's begin with a special case, where all the angles are right angles.

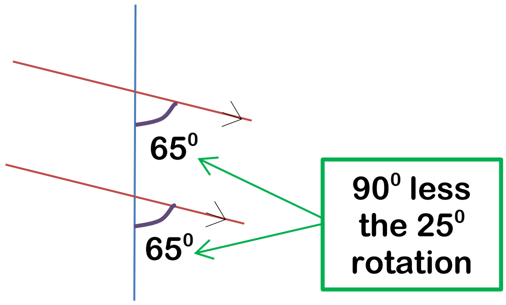


So we have two parallel red lines crossing a blue line at right angles. Remember, the arrowheads on the red lines represent that they are parallel. The F rule says that the corresponding angles, that's the inner angles of an F-shape, on the corresponding side of the blue line, are the same.



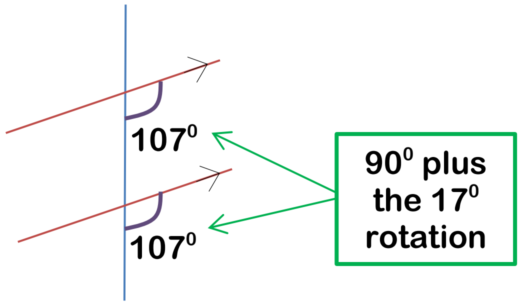
Here they are both right angles, which means that in this special case they are both the same.

Now if we rotate the top red line clockwise by 25^0 (about the point where it crosses the blue line) then to keep the two red lines parallel we have to rotate the bottom red line clockwise by 25^0 too.



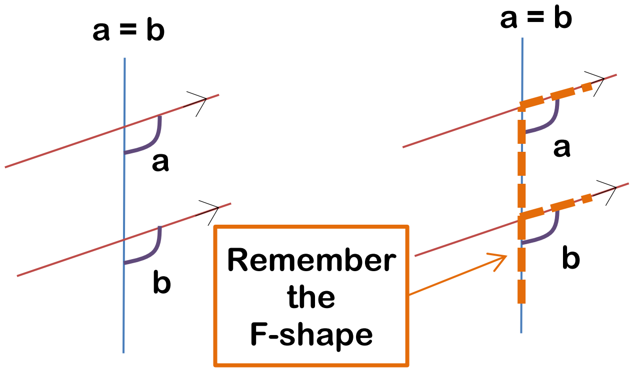
So the two angles inside the F-shape are still the same, 65^0. Because they both started as right angles, and by being rotated each by 25^0 clockwise ended up 25^0 less than 90^0, which is 65^0.

If we start again with our right angles and turn the red lines 17 anti-clockwise.

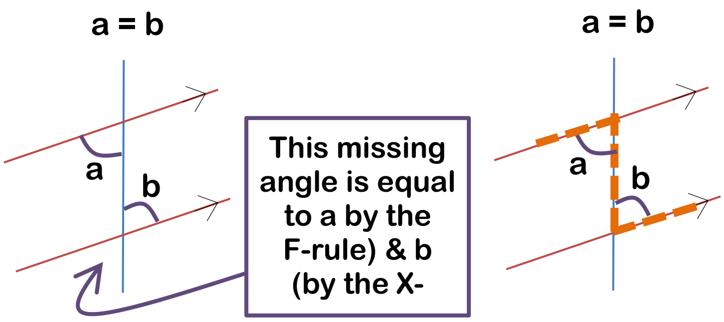


As long as we are keeping the two red lines parallel, whatever angle we rotate them both, we end up with 90^0 plus that amount (anti-clockwise turn) or minus that amount (clockwise turn) so the two angles are always the same.

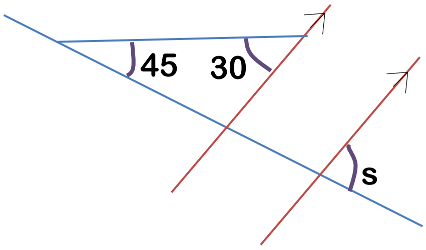
So for any two parallel lines crossing a third line, the two angles on the corresponding side are always the same.



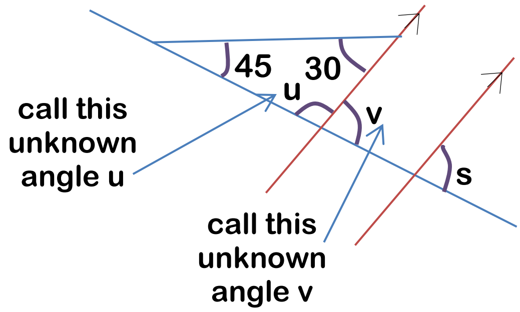
You can solve lots of different angle questions with these two rules. There are a couple more rules, but they are really just extensions of these two, like the alternating angles rule, known as the Z-rule.



Let's look at a multi-step problem using this and previous angles ladder steps.



The only angles we have been given are the 45 and 30, and the angle we want to know is s. It is hard to see any link between the 45, the 30 and s, but there are some other missing angles (as yet unnamed) that we could find. We also have two parallel (red) lines, which might imply the use of the Z (alternating) or F (corresponding) angles rules, but again, not without some more information. Let's label a few more angles and see where we can get to!



So u is in a triangle with angles 45 and 30, so the three sum to 180, so u = 105.

v is on a straight line with u (which we now know is 105) so they also add up to 180. Hence v = 75.

s and v (which we now know is 75) are corresponding angles, they make an F shape with the two red parallel lines, so s = v = 75.

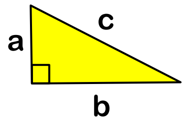
So by gradually filling in any missing angles we could, we found lots more information, and found s itself to be 75.

We could continue and find other missing angles withint this diagram, but as we have the angle we wanted, s = 75, we'll stop here.

Step 10 Pythagoras' Theorem.

In around 500BC lived a mathematician called Pythagoras. In some ways Pythagoras was an early feminist as he taught men and women together even though it was illegal in those times! This means it is likely that some women helped evolve the ideas behind Pythagoras' theorem, and it is quite possible that - as ideas are often named after the "group leader," not necesarryily the inventor - this theorem may have actually been discovered by a female mathematician.

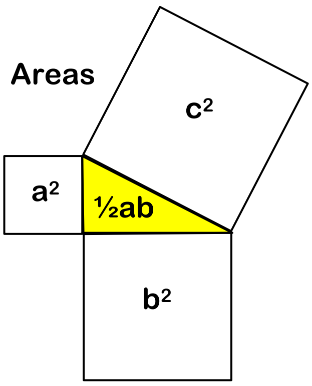
They studied right angled triangle, calling the short sides a and b, and long side (hypotenuse) c.



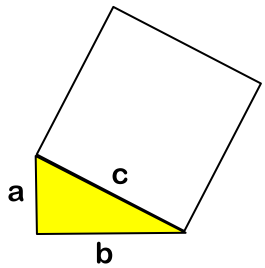
Pythagoras, or possibly one of his male, female or other gendered team discovered that for right angles triangles

$$a^2 + b^2 = c^2$$

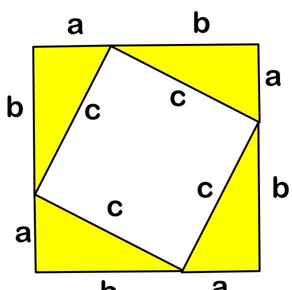
This can be thought of visually by drawing a square from each side, and seeing that the area of the two smaller squares together (a^2 and b^2) is the same as the area of the larger square (c^2).



Let's look at why this theorem is true.



If we take our c-square from the above diagram (with area c^2) and surround it with more of our a, b, c triangles, we can make a large square of side a + b.



There are two possible ways to find the area of this large a+b square. One is to multiply the two sides (a + b) and (a + b) together. The other is to take the area of the small middle c-square, and add the area of the four a,b,c triangles.

Area of 1 a,b,c triangle = ½ab
Area of 4 a,b,c triangles = 4x½ab = 2ab
Area of the c-square = c^2

Area of the large (a + b) square = $(a + b)^2$
= (a + b) (a + b)
= $a^2 + b^2 + ab + ab$
= $a^2 + b^2 + 2ab$

$(a$	$+\ b)$	
a^2	ab	$(a$
ab	b^2	$+b)$

Because these two ways of working give the same thing, they must be the same, or equal, and so can be written on each side of an equation.

c-square

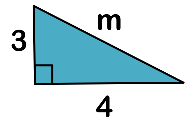
big (a + b) square

4 triangles

$$\underbrace{a^2 + b^2 + 2ab}_{\text{big }(a+b)\text{ square}} = \underbrace{c^2}_{\text{c-square}} + \underbrace{2ab}_{\text{4 triangles}}$$

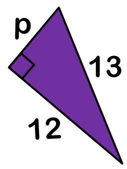
and if we subtract 2ab from both sides we get..

$$a^2 + b^2 = c^2$$



To find the hypotenuse m, we substitute m, 3 and 4 into Pythagoras' equation.

Equation	**Action**
$3^2 + 4^2 = m^2$	Simplify
$9 + 16 = m^2$	Sum
$25 = m^2$	$\sqrt{\ }$
$5 = m$	Switch sides to make m the subject
$m = 5$	



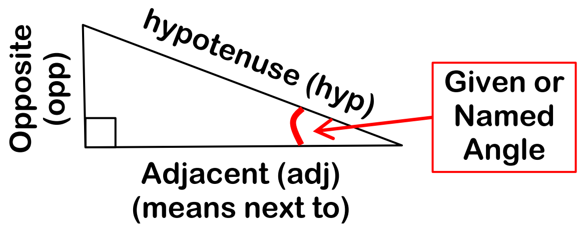
Equation	**Action**
$p^2 + 12^2 = 13^2$	Simplify
$p^2 + 144 = 169$	- 144

$p^2 = 64$	$\sqrt{\ }$
$p = 8$	

Note that with finding a short side, we have to subtract one of the squared constants to find the square of the missing side (and solve by $\sqrt{\ }$).

Step 11) Trigonometry

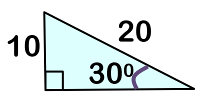
Trigonometry, often know as trig as an affectionate nickname, is a way of connecting two sides of a right angled triangle with one of the (non-right-angular) angles. The longest side is always opposite the right angle, and is always called the hypotenuse, or hyp for short. Its name can be remembered as being as big as a hippopotamus. For the shorter sides, when we have a given or named angle, the shorter sides are named by whether they are opposite (called opposite, or just opp for short) or next to (called adjacent, or adj for short).



We are already familiar with the name of the long side, or hypotenuse, but these two new names, opposite and adjacent (relative to a given or named angle) are reeeeeaaaaally important for understanding trig, so play around with a few right angled triangles with a given or named (non-right) angle, then name the three sides relative to that.

The amazing thing to know with trig is that for a given angle, the ratio between the opposite and the adjacent is always the same, however big (or tiny) the triangle is. The same is true for the ratio between the opposite and the hypotenuse, and between the adjacent and the hypotenuse.

The is because if you fix a right angle and a second angle, and as the 3 angles in a triangle have a fixed sum of 180^0, you have effectively fixed all three angles. This means that any triangle with those three angles is an enlargement (or an enshrinkment!) of any other triangle with those three angles. So the ratios between sides in those angles will always be the same. (It is worth reading the step on congruence and similarity here from the TREE ladder.



So for example if we have a right angled triangle with a given angle of 30^0, and the opposite 10cm, and hypotenuse 20cm, the ratio of opposite over hypotenuse is 0.5. This will be true for any other such triangle. Now this is all a bit of a mouthful, so mathematicians have given three names to the ratios of the three possible pairings of sides. We call them sine, cosine and tangent (or sin, cos and tan for short).

$$\sin(angle^0) = \frac{opp}{hyp}$$

$$\cos(angle^0) = \frac{adj}{hyp}$$

$$\tan(angle^0) = \frac{opp}{hyp}$$

This is often memorised simply using the word SOH-CAH-TOA (the first letter of each part of the related three triplets).

There are dozens of mnemonics for memorising these 9 letters too. Here are a couple of examples..

Some old Hags Can't Always Hide Their Old Age

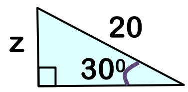
Some Of Her Children Are Having Trouble Over Alge bra

Let's look at sin for a little while, going back to our previous example, a right angled triangle with a given angle of 30^0, and the opposite 10cm, and hypotenuse 20cm. This means that

$$\sin(30^0) = \frac{10}{20} = \frac{1}{2} = 0.5$$

Now had we not know what the hypotenuse was, but known the angle was 30^0 and the opposite was 10, we could have worked out that the hypotenuse had to be 20, as this is the only length that would keep the sin ratio for 30^0, written sin(30) or just sin30, as ½ or 0.5.

$$\boxed{sin30 = \frac{1}{2}}$$

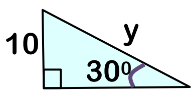


$$sin30 = \frac{z}{20}$$
$$0.5 = \frac{z}{20}$$
z=20 x 0.5 = 10

Similarly if we had not known the opposite but had known the hypotenuse was 20 and the angle was 30^0 then we could have deduced that the opposite was

10, as this was the only side length that would maintain the fact that sin30 = 0.5

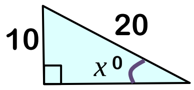
$$\boxed{sin30 = \frac{1}{2}}$$



$$sin30 = \frac{10}{y}$$
$$0.5 = \frac{10}{y}$$
$$y = \frac{10}{0.5} = 10 \times 2 = 20$$

Finally, had we known that the opposite was 10, and the hypotenuse was 20, but not known the given angle, if we knew that sin30 = 0.5 (which it always does and shortly we'll see two ways to find which angle goes with which ratio), then we could immediately know that the angle must have been 30^0.

$$sin30 = \frac{1}{2}$$



$$\sin(x) = \frac{10}{20}$$

$$\sin x = \frac{1}{2} = 0.5$$

$$x = 30$$

Having been through all three possibilities, we can see that if we know any two of the angle, the opposite or the hypotenuse, we can use $\sin(angle^0) = \frac{opp}{hyp}$ to find the third missing piece of information.

The fact that sin30 = 0.5 can be found out in two different ways. We can use a table of trig ratios

Angle	sin(e)	cos(ine)	tan(gent)
0	sin0 = 0	cos0 = 1	tan0 = 0
5	sin5 = 0.087	cos5 = 0.996	tan5 = 0.087
10	sin10 = 0.174	cos10 = 0.985	tan10 = 0.176
15	sin15 = 0.259	cos15 = 0.966	tan15 = 0.268
20	sin20 = 0.342	cos20 = 0.94	tan20 = 0.364
25	sin25 = 0.423	cos25 = 0.906	tan25 = 0.466
30	sin30 = 0.5	cos30 = 0.866	tan30 = 0.577
35	sin35 = 0.574	cos35 = 0.819	tan35 = 0.7
40	sin40 = 0.643	cos40 = 0.766	tan40 = 0.839
45	sin45 = 0.707	cos45 = 0.707	tan45 = 1
50	sin50 = 0.766	cos50 = 0.643	tan50 = 1.192
55	sin55 = 0.819	cos55 = 0.574	tan55 = 1.428
60	sin60 = 0.866	cos60 = 0.5	tan60 = 1.732
65	sin65 = 0.906	cos65 = 0.423	tan65 = 2.145
70	sin70 = 0.94	cos70 = 0.342	tan70 = 2.747
75	sin75 = 0.966	cos75 = 0.259	tan75 = 3.732
80	sin80 = 0.985	cos80 = 0.174	tan80 = 5.671
85	sin85 = 0.996	cos85 = 0.087	tan85 = 11.43
90	sin90 = 1	cos90 = 0	tan 90 = ???

From this table you could quickly establish that sin30 = 0.5, and that cos80 = 0.174, or that if you have a tan ratio between the opp and the adj of 2.145, that the angle is 65^0.

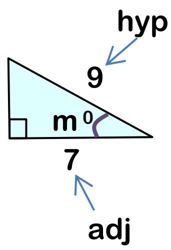
Unfortunately this table, which is already quite big, only has every 5^0, what if you wanted to work with an angle of 37^0, or if you had angles that were measured to 2 decimal places?

For many years, mathematicians used huge trig tables to look up the ratios that they needed, sin, cos or tan. But nowadays it has got a lot easier.

If you type sin30 into a calculator it gives you the ratio of 0.5. Similarly if you type in cos80 it gives you 0.1736481777. This is 0.174 (like in our table above) when rounded to 3dp, but the calculator will give you your trig ratios to several dp, from which you can then round your final answer to whichever degree of accuracy you require. Similarly if you know the ratio because you know the two sides, you can then use the $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ buttons to find the angles.

If you type in $\sin^{-1}(1/2)$ to your calculator it will give you the ratio of 0.5.

Let's look at a couple of examples…



$$\cos(m) = \frac{adj}{hyp}$$

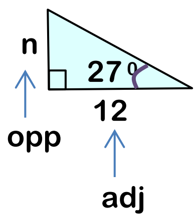
$$\cos(m) = \frac{7}{9}$$

$$m = cos^{-1}(7 \div 9)$$

m = 51.05755873

A calculator gives lots of decimal places but we can round it to whatever is suitable.

m = 51.1 (3sf)

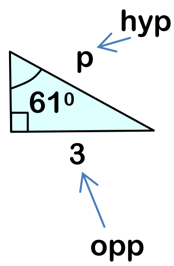


$$\tan(angle) = \frac{opp}{adj}$$

$$\tan(27) = \frac{n}{12}$$

$$n = 12\tan(27)$$

$$n = 6.11 \text{ (3sf)}$$



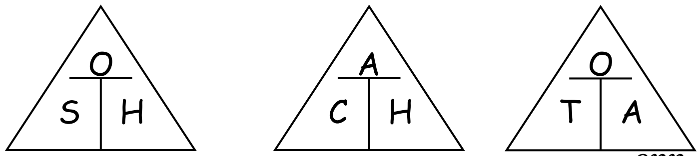
$$\sin(angle) = \frac{opp}{hyp}$$

$$\sin(61) = \frac{3}{p}$$

$$p\sin(61) = 3$$

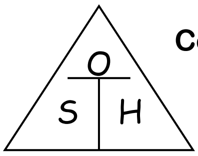
$$p = \frac{3}{\sin(61)}$$

Notice on these last two questions, on one you got $12\tan(27)$ (times) and on the other you got $\frac{3}{\sin(61)}$ (divide). You can use the trig triangles to save rearranging equations, to tell you instantly whether to times or divide.



$sinx = \frac{opp}{hyp}$ $\quad cosx = \frac{adj}{hyp}$ $\quad tanx = \frac{opp}{adj}$

For example with the sin triangle.

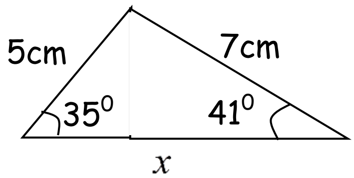


Covering opp **we get** $opp = hyp \times sinx$

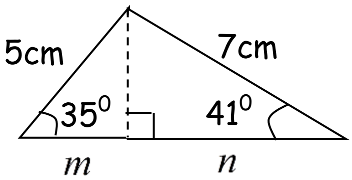
Covering hyp **we get** $hyp = \frac{opp}{sinx}$

Covering sin(x) we get $sinx = \frac{opp}{hyp}$ **so** $x = \sin^{-1}\left(\frac{opp}{hyp}\right)$

Finally we can sometimes apply trigonometry to non-right angled triangles like this one…



But we can split this into two right angled triangles



And now $x = m + n$

$m = 5cos35 = 4.09576$

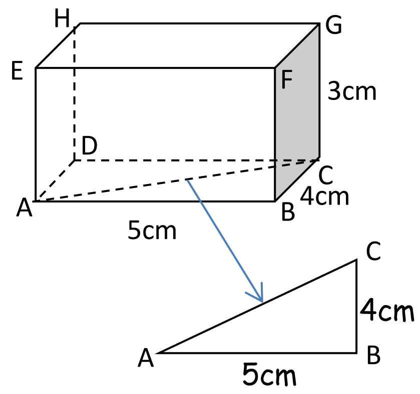
$n = 7cos41 = 5.28297$

$x = m + n = 9.4\ (1dp)$

3D Pythagoras and Trigonometry

Using Pythagoras and trigonometry with 3D shapes is not a special new version of pythag and trig. Each triangle we use can still be thought of as a 2D triangle usually on the face of a 3D shape. The skill here is being able to sketch the needed 2D triangles from the 3D shape, to find the missing sides or angles.

Let's look at cuboid ABCDEFGH, and try And Find sides AC and AG, and angle CAG.

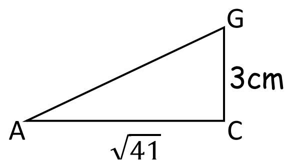


If we draw a diagonal (dashed) line across AC, we can see that AC is the hypotenuse of triangle ABC.

We can then use Pythagoras.

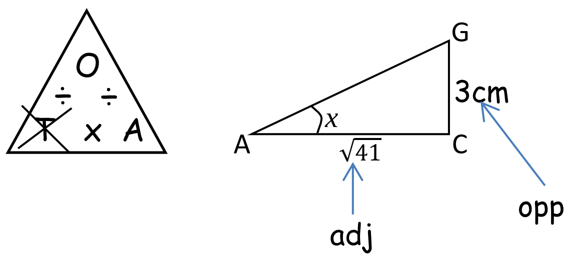
Equation	Action
$(AC)^2 = 4^2 + 5^2$	Square RHS
$(AC)^2 = 16 + 25$	Sum RHS
$(AC)^2 = 41$	$\sqrt{\ }$
$AC = \sqrt{41}$	

Now to find AG, we need the line AC across the base of the cuboid, the line CG up the side, and then the 3D diagonal AG. This makes triangle ACG. CG was given in the question (3cm) and we just worked out that AC was $\sqrt{41}$.



Equation	Action
$(AG)^2 = (\sqrt{41})^2 + 3^2$	Square RHS
$(AG)^2 = 41 + 9$	Sum RHS
$(AG)^2 = 50$	$\sqrt{\ }$
$AG = \sqrt{50}$	

Finally lets look at the angle CAG. We already have a sketch of this triangle from finding AG. Using this sketch we can label CAG as x.



We know the opp and the adj so we need to use tan.

$$tanx = \frac{3}{\sqrt{41}}$$

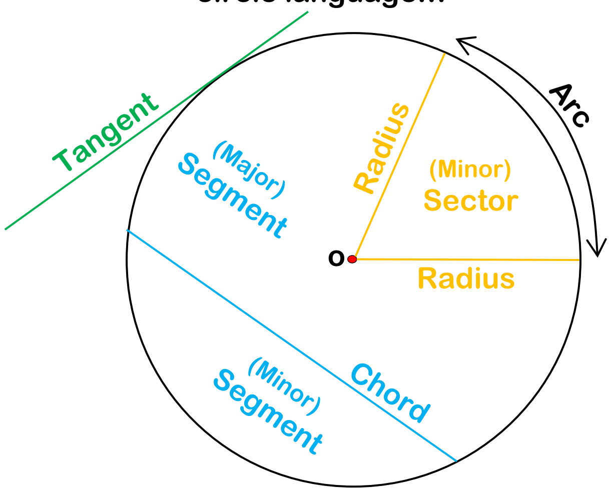
$$x = tan^{-1}\left(\frac{3}{\sqrt{41}}\right)$$

$$x = CAG = 25.1\ (3sf)$$

The interesting thing is that if we had been asked to find angle CAG from the start, we would still have had to find AC first. Similarly if we had been asked to find side AG, we would first need to find AC. So 3D pythag and trig questions are often multi-step, using one 2D triangle to find a first diagonal, and with that info using a second step of pythag or trig to find a second diagonal.

Step 13) Circle Theorems

There are several circle theorems. They are all facts to do with angles made by lines connecting points at the centre or circumference of a circle in various ways. Firstly you need to know some important circle language…



Radius: Is the line (or its length) from the centre of a circle to a point on its circumference (Plural is radii).

Tangent: Touches a curve at exactly one point.

Chord: Is the line between two different points on the circumference of a circle.

Segment: Is one of the two areas created when a circle is separated by a chord.

Sector: Is the slice (shaped like a pizza slice) when a circle is cut into two pieces using two radii.

Arc: An arc is a part of the circumference of a circle between two points.

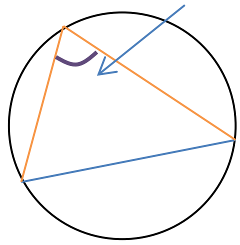
Major/Minor: The major segment/sector/arc is the larger part. The minor segment/sector/arc is the smaller part.

Remember that a circumference can be thought of as two radii in a straight line, or as a special chord that goes through the centre (or origin) of a circle.

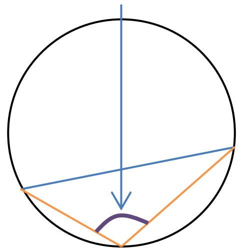
We will approach the circle theorems by asking you to investigate and get a feel for each theorem.

Let's start by learning the phrase subtend an angle from a chord. This is where we draw two straight (orange) lines from either end of a chord (blue), to a point on either the major or the minor arc.

Subtend an angle in the major arc

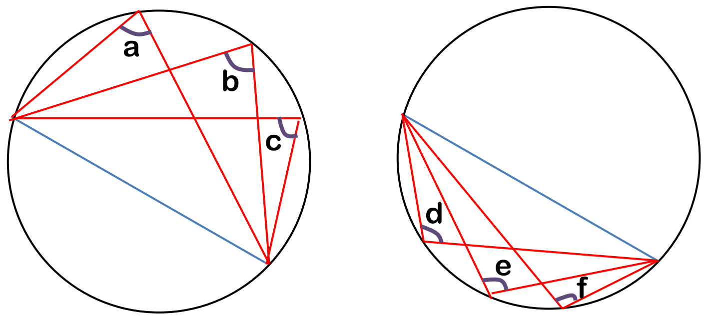


Subtend an angle in the minor arc



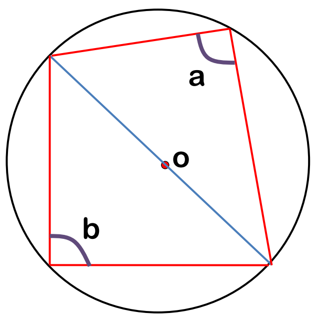
Let's look at some ideas and investigate them. Remember if angles are within 1^0 or 2^0 of each other it is worth considering that they might be the same, and that the difference might be due to drawing or measuring inaccuracies.

Rule 1. Investigating angles subtended on the major arc and the minor arc of the same chord.



Draw some circles like those above and subtend the chord to several points on the major arc, see if there is a relationship between "your," a, b & c. Do the same for "your," d, e & f on the minor arc. If you find a rule 1 here, what would you call it?

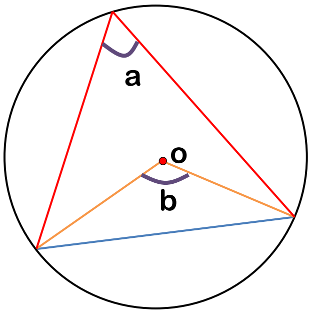
Rule 2. Investigating the angle from a diameter.



Please note that the major and minor arc are the same size here, but investigate a few angles subtended on either side never the less! If you can find a rule 2 here then what would you name it?

Rule 3. Investigate the relationship between the angle at the centre and the angle at the arc.

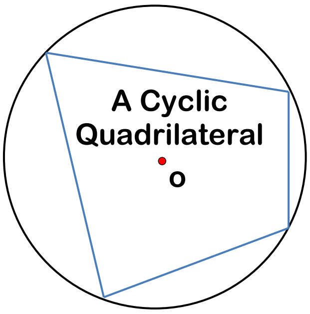
Draw few like this and see if you can find a relationship between "your," a angles (at the arc) and "your," b angles (at the centre).

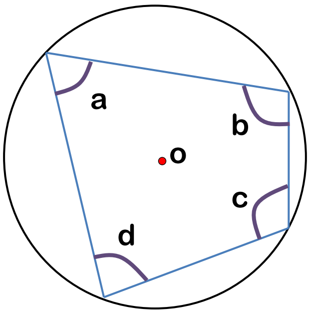


If you can find a rule 3 what would you describe it as?

Rule 4. Angles in a cyclic quadrilateral.

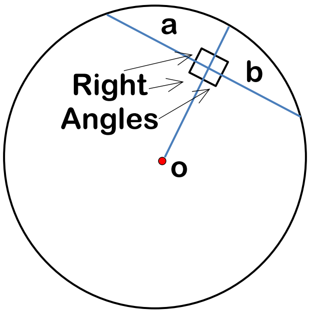
A ***cyclic quadrilateral*** is a 4-sided polygon, with all four vertices on the circumference of a circle.





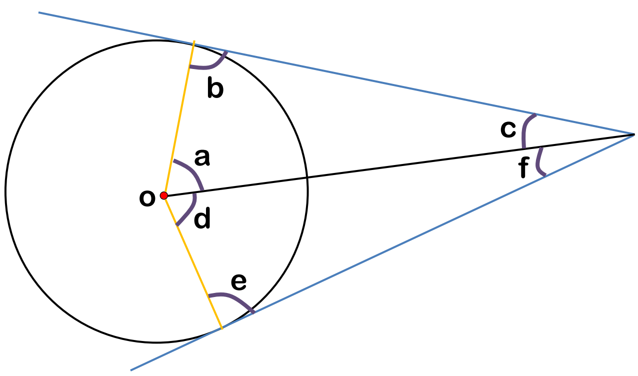
Can you spot a relationship between "your," a, b, c, & d? If so what would you call this rule 4?

Rule 5. Investigate a chord being cut by the perpendicular radius.



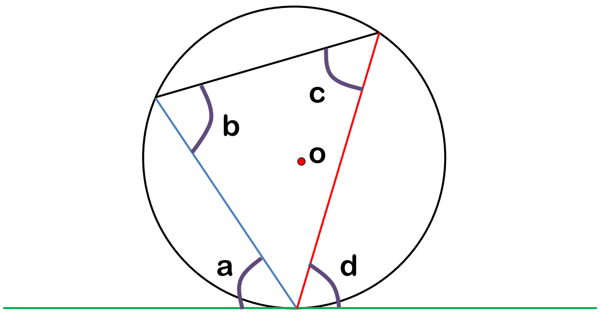
Measure "your," lengths a and b on either side of the cut chord, what do you notice. If you spot a rule 5 here give it a name.

Rule 6. Let's investigate the triangles created by two tangents, and the centre of the circle.



Can you see any relationship(s) between "your," angles a, b, c, d, e, & f? If so how would you describe this relationship, what would you call this rule 6?

Rule 7. Last but not least! Let's trying subtending a (black) chord and drawing a (green) tangent at this point on the circumference.



Look at "your," angles a, b, c & d, a & d are the angles between the (red and blue) chords and the (green) tangent. c & d are the angle between the red and blue chords, and the black chord they were used to subtend. Can you spot a rule 7 here? If so give it a name.

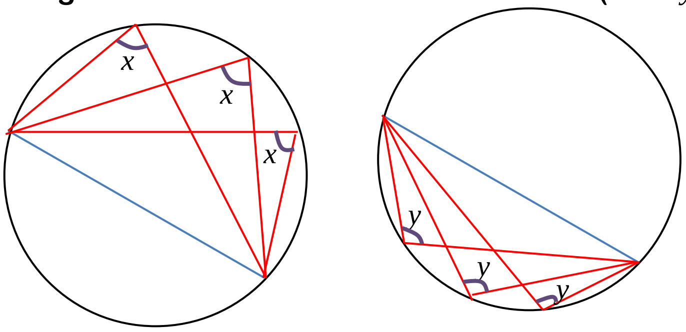
Having done that, let's make a summary sheet of all the rules you may or may not have discovered in your investigation. The names of the circle theorems have not been universally agreed by mathematicians (except rule 7). The bold names at the top are yogi Toby's best name for quickly and catchily remembering the rules and their meanings. If you prefer the name you chose, or can think of a better one, feel free to use that!

The next page can be used as a single printed sheet, when initially working with circle theorems (untill they are all embedded in your mind).

Circle Theroems Summary Sheet

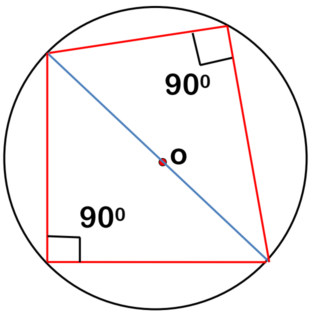
Rule 1. **Same Arc Same Angle**

The angles subtended from (the same chord) on the major arc are all the same (here labelled x). Similarly the angles on the minor arc are all the same (here y).



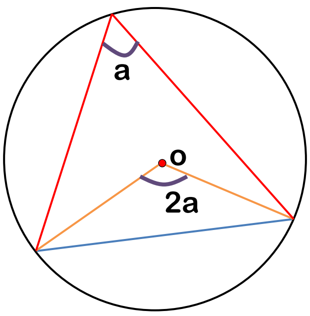
Rule 2. **Diameter gives a Right Angle**

When subtending a diameter, you always get a right angle (in either arc – the major and the minor arcs are the same here).

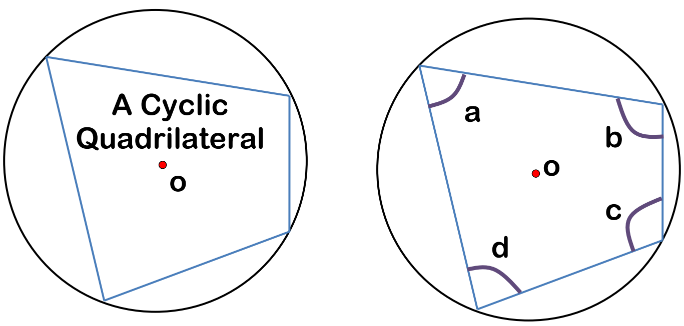


Rule 3. **Centre is Double the Circumference**

The angle subtended at the centre of a circle, is double the angle subtended at the circumference.

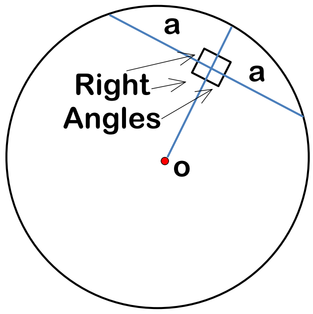


Rule 4. **In a cyclic quadrilateral opposite angles have sum 180^0**

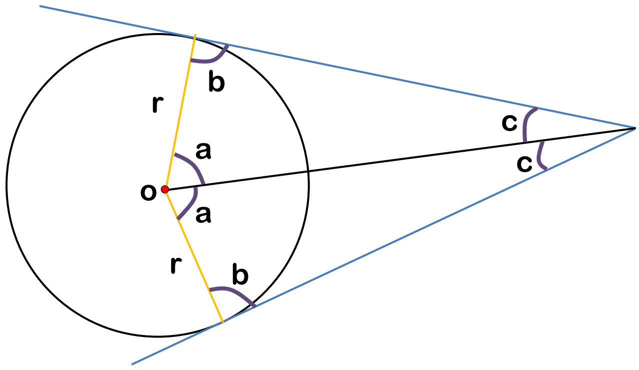


a + c = 180^0 and b + d = 180^0

Rule 5. A radii at right angles to a chord, bisects that chord.



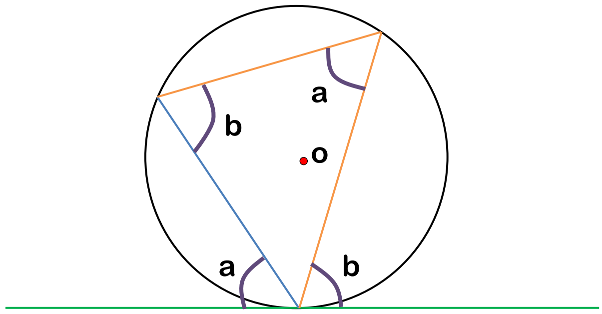
Rule 6. **Two tangents and the two radii that intersect them form two congruent triangles.**



Rule 7. **The Alternate Segment Theorem**

(This is the only circle theorem with a proper agreed name).

The angle in the minor arc between the blue chord and the green tangent (a) is the same as the angle subtended by the orange lines on the major arc of the blue line (also a).



<u>Can you mentally visualise the rules from this list?</u>

Rule 1. **Same Arc Same Angle**

Rule 2. **Diameter gives a Right Angle**

Rule 3. **Centre is Double the Circumference**

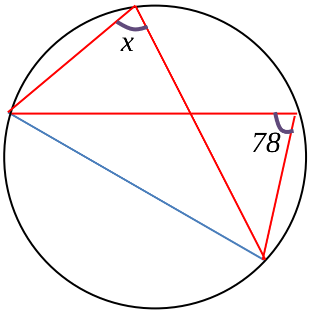
Rule 4. **In a cyclic quadrilateral opposite angles have sum 180^0**

Rule 5. **A radii at right angles to a chord, bisects that chord.**

Rule 6. **Two tangents and the two radii that intersect them form two congruent triangles.**

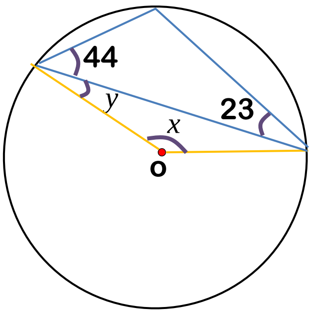
Rule 7. **The Alternate Segment Theorem**

For each rule we can ask a basic question such as find angle x...



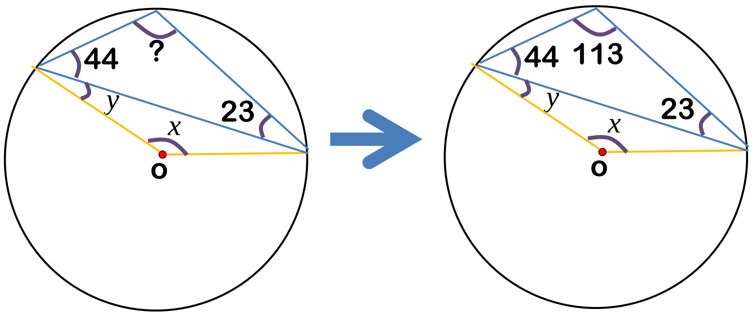
The answer is that $x = 78$, because angles subtended from the same chord in the same arc are the same. We can ask questions like this for each, but this is really a memory test, for memorising how the rules look, and then stating the name of the rule as the reason.

But the real skill in this step is in solving multi-step problems, where we have to apply more than one circle theorem, and possible some other angle rules too!

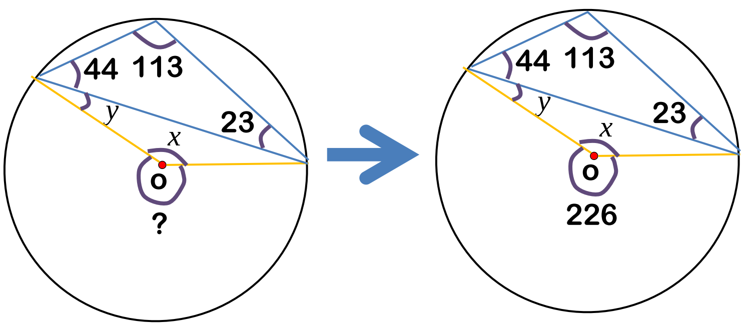


This is a great question. It appears at first glance that it is impossible to solve this question using a circle theorem, despite the obvious presence of a circle. But we can find the angle at the top, as it is in a triangle with two known angles, 23 and 44.

180 – (23+44) = 113.

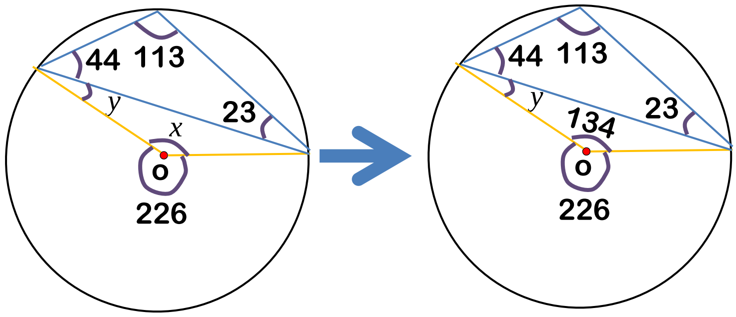


Then the angle at the centre must be double the angle at the circumference, and double 113 is 226.



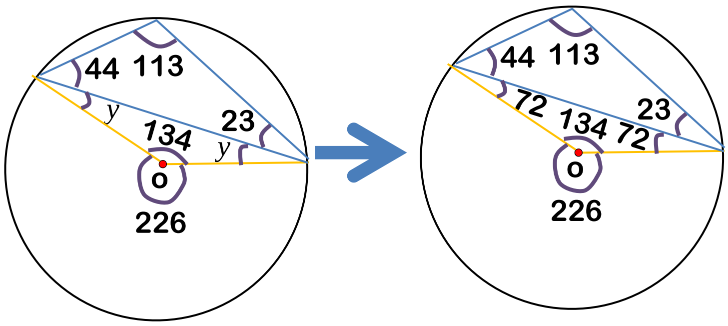
Now x and 226 form a full turn of 360

So $x = 360 - 226 = 134$



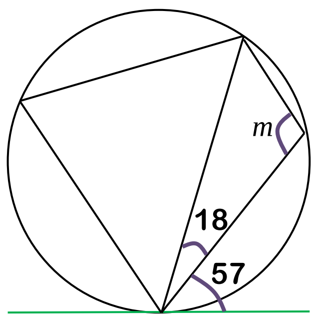
And finally the triangle with our unknown angle y is isosceles because it has two radii from the same circle, so they are the same. Hence the triangle has two acute angle ys and a 226.

Equation	*Do*
$2y + 226 = 370$	*(-226)*
$2y = 144$	*(÷2)*
$y = 72$	



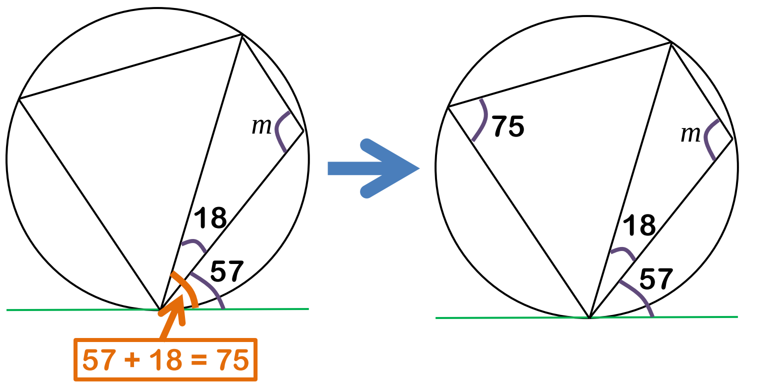
Notice that the only circle theorem we have used here is the angle at the centre being double at the circumference rule. This is often the case, one or two circle theorems combined with many other angle rules can make a long and complex question like this. Just keep filling in missing angles you can find, until you are able to find the ones you want to.

Let's try a question that needs two circle theorems.

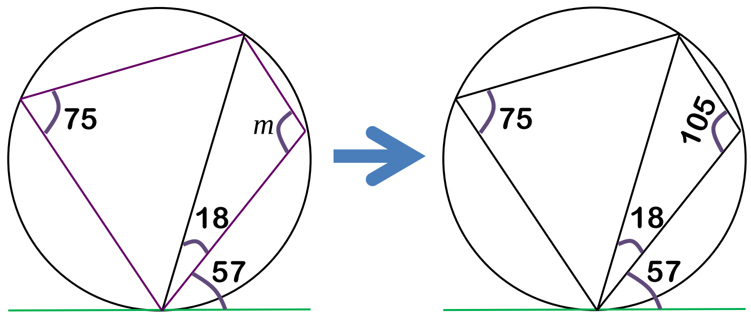


m and the angle of 18 are in a triangle, and m is also in a cyclic quadrilateral but we don't currently know any of the other angles in that cuclic quadrilateral. However, the 18^0 and 57^0 angles together make a 75^0 angle which is between a chord and a tangent.

Hence by the alternate angles theorem, the angle on the alternate segment must also be 75⁰.

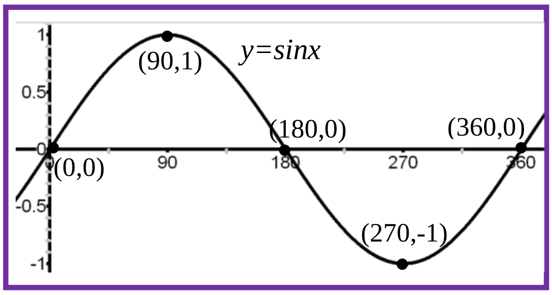


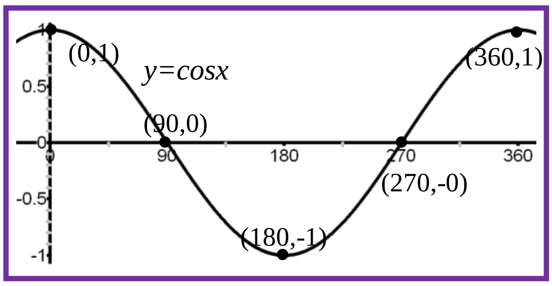
Now m and the 75⁰ angle are opposite each other in a (purple) cyclic quadrilateral which means their sum is 180⁰. So $m = 180 - 75 = 105$

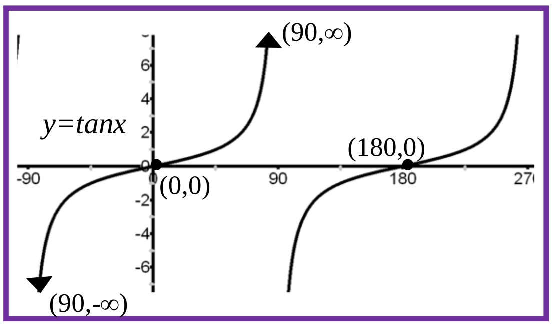


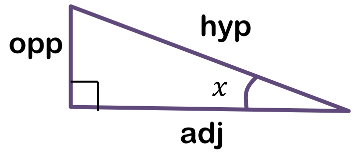
Step 14) Trig Graphs

It is possible to plot a graph for trig functions, with the angle along the x axis, and the sine, cosine or tangent values on the y axis. In other words we plot x against $sinx$, $cosx$, or $tanx$. The graphs have shapes that repeat every 360⁰ (or 180⁰ in the case of $tanx$). Here's how they look...









$sinx = \frac{opp}{hyp}$ & $cosx = \frac{adj}{hyp}$ & $tanx = \frac{opp}{adj}$

Let's look at $sin0$, $sin90$ & $sin180$,

When x approaches 0, the opp approaches 0 too, which means when you divide by the hyp it will make 0. This gives us the point (0,0). When x approaches 90, the opp approaches being the same length as the hyp, hence when x is 90, the opp and the yp are the same length, so $sin90 = 1$giving us point (90,1). When x goes all the way to the other side (East!) then the opp will again approach zero.

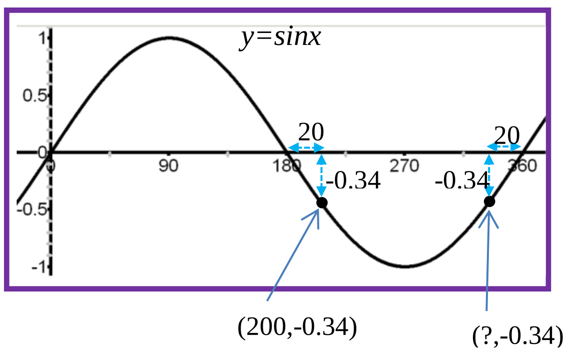
We can do a similar thing to understand these values for $cosx$.

In the case of $tanx$ it is the opp that approaches infinity, but the adj stays a finite number, so this graph is not bound between 1 and -1 as the $sinx$ & $cosx$ graphs are.

The final thing to understand here, is that the trig values for 360 are all the same as for 0, because the angles 0 and 360 are in the same direction, you have the same triangle. So 1 is the same as 362, 2 is the same as 362 and so on. The graphs all repeat.

For a given y value, the curves have two x values between 0 and 360. And they have another two between 360 and 720, and so on ad infinite repetitions of 360.

The value of $sin200 = -0.34$. This means that when $x = 200$, then $sinx = -0.34$. We can use the graph to see that there is another value where $sinx = -0.34$ and that it is the same distance left of (360, 0) asour first solution is right of (180, 0). So this point is (360-20, 0) which is (340, -0.34).



Because this graph repeats with period 360, there must be another two points where $sinx = -0.34$ between 360 and 720, and because the period of the graph is 360, they must be exactly 360 on from the points we just found.

So the equivalent point 360 right of (200,-0.34) will be (200 + 360, -.034) which is (560, -0.34),

And the equivalent point 360 right of (340, -0.34), will be (340 + 360, -0.34) which is (700, -0.34)

So we now have 4 soluitions of the equation $sinx = -0.34$ in tha range $0 \leq x \leq 720$ and these are

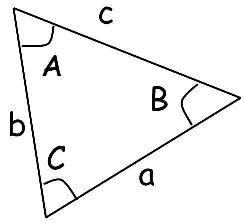
x = 200, 340, 560 & 700.

In fact there are infinitely many solutions to this equation (200 + any multiple of 360, -0.34)

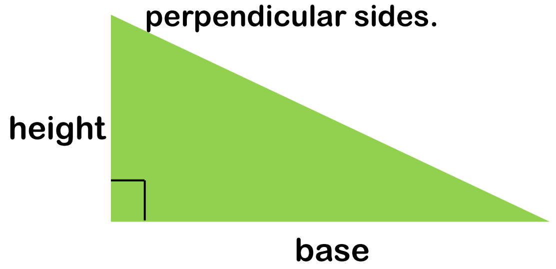
and (340 + any multiple of 360, -0.34)

Step 15) Sin, Cos & Area Rules.

For this step we will be working with triangles with the vertices labelled (little) a, b & c, and the angles opposite each side labelled (capital) A, B & C like so…

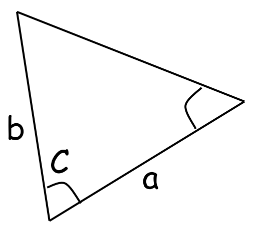


The area of a right angled triangle, where the two perpendicular sides are thought of as being the base and the height, is half of the product of the perpendicular sides.



$$Area = ½ \times base \times height$$

Now if we increase or decrease the right angle to make it say 100 (an increase of 10) or 80 (a decrease of 10), the base will be the same, the side previously representing the height will be the same, but the actual height perpendicular to the base will have reduced a bit. If we exaggerate this action and take the angle to either 0 or 180, then the triangle will become flat. Now the actual area rule for a triangle is this…



$$Area = \frac{1}{2}absinC$$

Now looking at our example before of the right angled triangle, here C = 90.

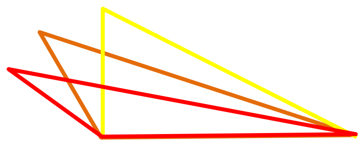
So the area is $\frac{1}{2}absin90$

and as $sin90 = 1$, then the area is just $\frac{1}{2}ab$

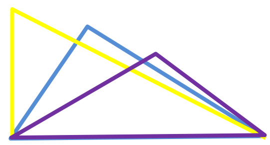
which is just the area of a triangle formula. Similarly

$sin0 = sin180 = 0$ so when the angle is 0 or 180 the area is 0. This is an interesting way to think about the sine rule, that with a right angled triangle it gives you the area of a triangle formula, and as the height gets less by changing the angle the sin of the angle slowly reduces the area until the height is zero as the area is zero at angles 0 or 180. But it doesn't actually prove that the formula reduces the area at the right speed.

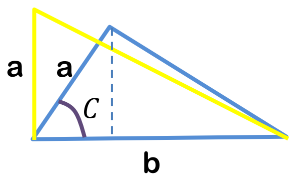
So as we increase the angle from a right angle in the yellow triangle to a larger angle in the orange and a larger still in the red triangle (but not yet 180^0) the area decreases.



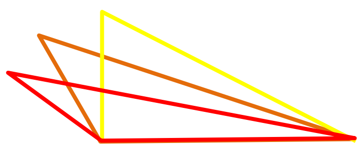
And similarly if we increase the angle to the blue, then the purple triangle the area will get less.



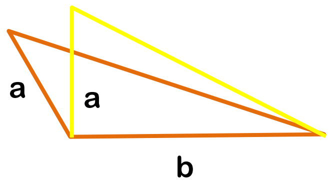
Let's look at the example where we reduce the angle. The yellow triangle is right angled with side a and base b. So the diagonal of the blue triangle is also a as it came from rotating the vertical side of the yellow triangle.

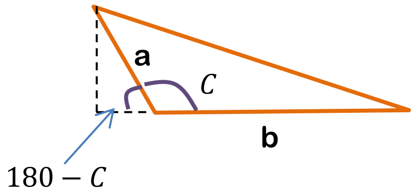


Now a on the blue triangle forms the hypotenuse of a right angled triangle who's opposite angle relative to x is the height of the blue triangle. So this height is $asinC$. So the height of the blue triangle is $asinC$ and the base is b, so the area is $absinC$.



This also works when we increase the right angle.

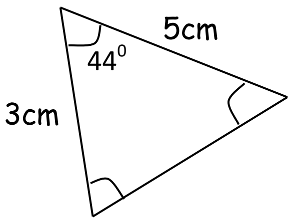




So the height of the orange rectangle is $\operatorname{asin}(180 - C)$, but we saw in the last step on sin graphs that

$\sin(180 - C) = \sin(C)$, so the height of this shape is $\operatorname{asin}(C)$ and the area is $\operatorname{absin}(C)$ as required.

Let's try and find the area here…

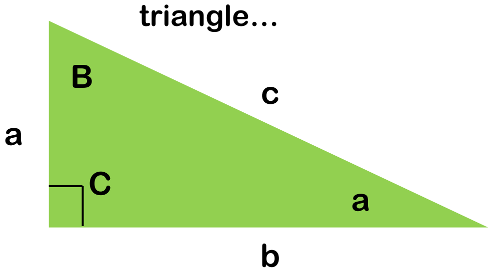


$$Area = ½ \times 35sin44$$

$$= 5.21\ (3sf)$$

Next is the sine rule.

It is quite easy to show this using a right angled triangle…



Using the sine ratio

$sinA = \frac{a}{b}$ and $sinB = \frac{b}{c}$

Rearranging both we get

$c = \frac{a}{sinA}$ and $C = \frac{b}{sinB}$

as these both equal C

$$\frac{a}{sinA} = \frac{b}{sinB}$$

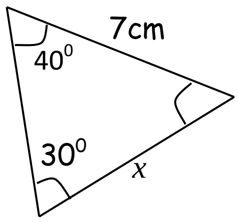
and this can be rearranged to give

$$\frac{sina}{A} = \frac{sinb}{B}$$

By extension

$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$ and $\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$

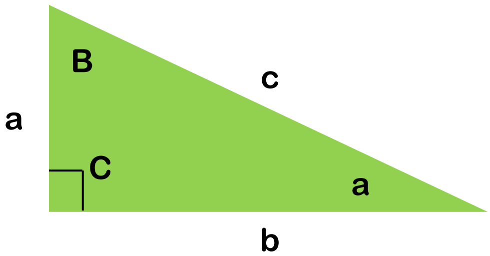
In other words the ratio between a side and the sine of the angle it is opposite is always the same for all the sides and respective angles in a given triangle.



So $\frac{x}{sin40} = \frac{7}{sin30}$

$$x = \frac{7sin40}{sin30} = 9.0(1dp)$$

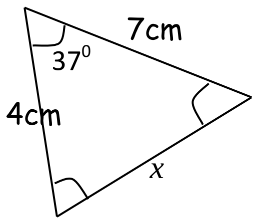
Just as the area rule can be thought of as an extension of the area of a triangle for non-right angled tirangles, the cos rule can be thought of as an extension of Pythagoras for non-right angled triangles. With a right angled triangle, Pythagoras states that $a^2 + b^2 = c^2$ but when the right angle gets bigger, then c (opposite the "right angle") increases, and when it gets smaller it decreases.



Increasing this angle will **increase** c
Decreasing this angle will **decrease** c

$$c^2 = a^2 + b^2 - 2abcosC$$

When c=90, cosC = 0, and we just get pythag
When C<90, cosC>0, and so c is reduced
When C<90, cosC<0, and so c is increased



$$x^2 = 4^2 + 7^2 - 2 \times 4 \times 7cos37$$

$$x^2 = 28 + 49 - 56cos37$$

$$x = \sqrt{20.2764}$$

$$x = 4.5\ (1dp)$$