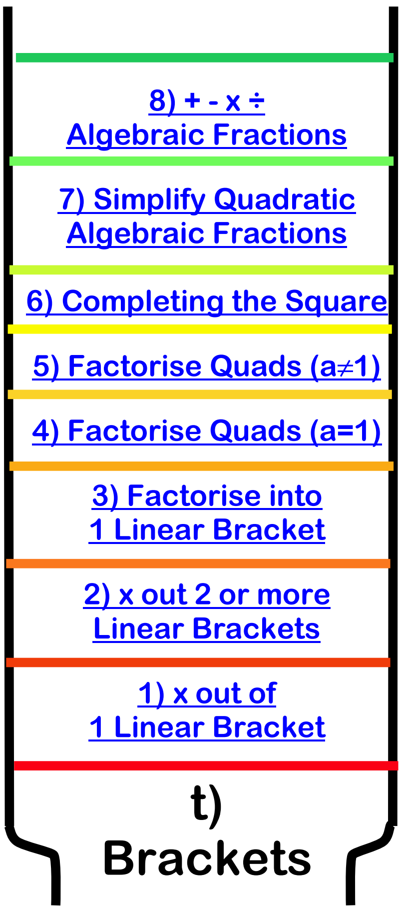
t) Brackets.



Step 1) x out of 1 Linear Bracket

What happens if we take several of a given linear expression?

What if we have 3 of $x+2$?

We simply write this as $3(x+2)$
Just as $5x$ means $5 \times x$
Similarly $3(x+2)$ means $3 \times (x+2)$

Now the question is, can we simplify this expression into one without brackets?

Well on one level we can just take 3 lots of the bracket... literally!

$$\begin{aligned} &3(x+2) \\ &= (x+2)+(x+2)+(x+2) \\ &= (x+x+x)+(2+2+2) \\ &= 3x+6 \end{aligned}$$

Amazingly

$$3(x+2) \equiv 3x+6$$

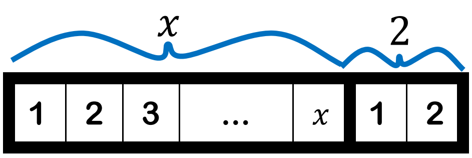
They are always the same!

The symbol $\equiv$ means two expressions are equivalent, or always the same whatever values of x (or any other variables) you substitute in.

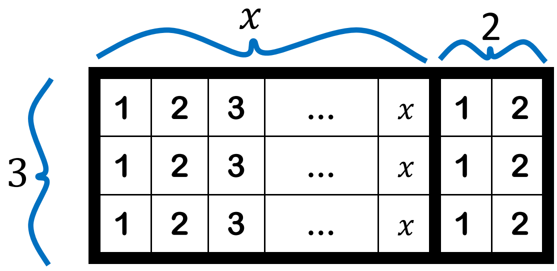
Let's try a few values of x to see if this is really true.

	$3(x+2)$	$3x+6$	Same?
$x=1$	$=3\times3$ $=9$	$=3+6$ $=9$	Yes
$x=2$	$=3\times4$ $=12$	$=6+6$ $=12$	Yes
$x=4$	$=3\times6$ $=18$	$=12+6$ $=18$	Yes
$x=7$	$=3\times9$ $=27$	$=21+6$ $=27$	Yes

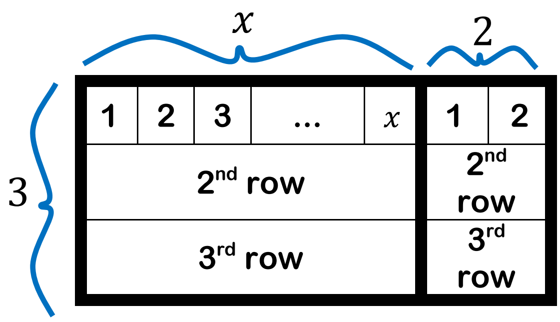
Can we explain this in a more general way? Let's use our grid of squares picture again. First we can represent $x+2$ in a grid, with x squares on the left, & 2 squares on the right



If we take three lots of this we get



or



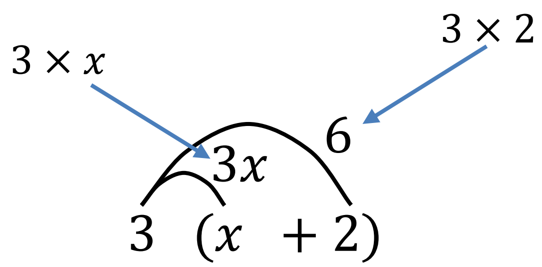
So on the left you have 3 lots of x which is $3x$, and on the right you have 3 lots of 2, which is 6.

We can simplify this in a grid method...

$3 \times x$ $\qquad\qquad$ 3×2

$(x$	$+ \quad 2)$	
$3x$	6	3

And the claw method will also work.



Step 2: x out of 2 or more Linear Brackets.

Just as we can extend a grid method for multiplying out numbers of many digits, so we can for multiplying out expressions of many terms.
Let's look at $(x+3)(x+2)$

x					3			
1	2	3	...	x	1	2	3	x
2nd row of x					2nd row of 3			
3rd row of x					3rd row of 3			
. . .					. . .			
x^{th} row of x					x^{th} row of 3			
1	2	3	...	x	1	2	3	2
2nd row of x					2nd row of 3			

And this can again be simplified in a sketch grid.

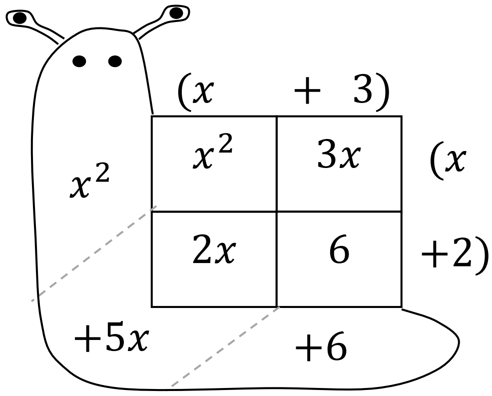
$(x$	$+ \quad 3)$	
x^2	$3x$	$(x$
$2x$	6	$+2)$

$$= x^2 + 3x + 2x + 6$$
$$= x^2 + 5x + 6$$

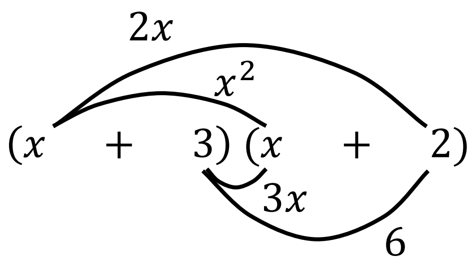
Note:
$3x + 2x \equiv 5x$

Building on the sketch grid, wonderfully, most of the methods we might use for doing 34×57, or to put that another way $(30+4)(50+7)$, can be used for multiplying out $(x+3)(x+2)$.

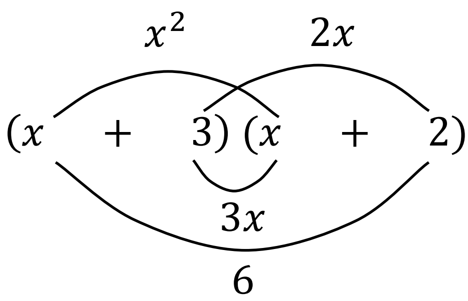
The Speedy Snail.



The Claw Method.



The Smiley Face Method.



The Foil Method

$(x+3)(x+2)$

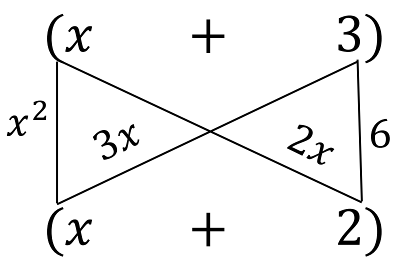
First $= x \times x = x^2$

Outside $= 2 \times x = 2x$

Inside $= 3 \times x = 3x$

Last $= 3 \times 2 = 6$

The Bow Tie Method



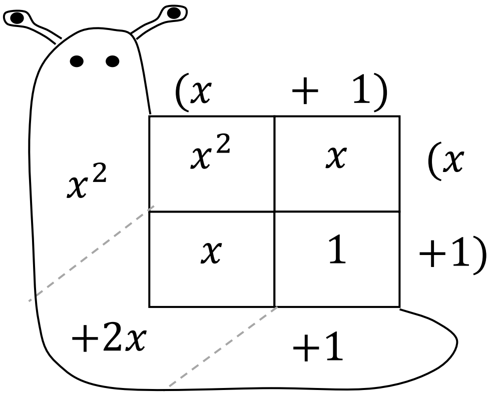
It is useful to see that more complex expressions can be multiplied out in the same way. Firstly we'll look at $(2x+5)(3x-1)$

The fact that like terms end up in diagonally aligned squares of the grid, becomes even more useful when we multiply larger expressions.

Let's look at $(x+1)^2(x+5)$

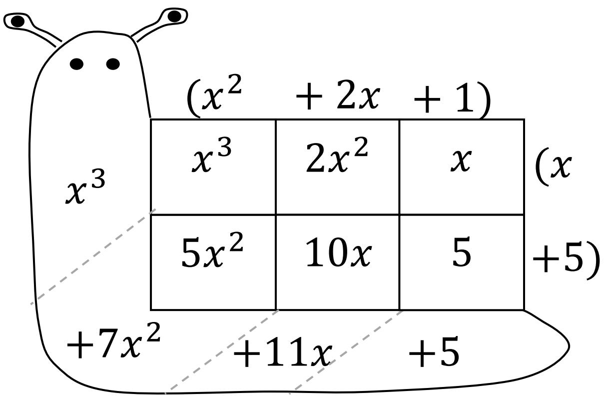
This is the same as $(x+1)(x+1)(x+5)$

We'll do the $(x+1)^2$ or $(x+1)(x+1)$ first.



So $(x+1)^2(x+5)$
$= (x^2+2x+1)(x+5)$

Now we have a quadratic expression multiplied by a linear expression. We can extent our snail grid, just like we did for increasing from multiplying 2-digit numbers to 3-digit numbers.



$$= x^3 + 7x^2 + 11x + 5$$

We can even write this using just the coefficients...

$$= x^3 + 7x^2 + 11x + 5$$

There are two types of quadratics that it is really important to know about. These are perfect squares (or complete squares with external constant 0) and difference of squares.

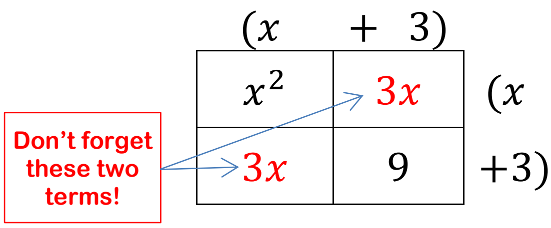
A **perfect square** is the result of squaring a linear bracket.

$(x+3)^2 \equiv x^2 + 6x + 9$
or
$(2x-1)^2 \equiv 4x^2 - 4x + 1$

So $x^2 + 6x + 9$, which can also be written $(x+3)^2$ Is a perfect square, as is $4x^2 - 4x + 1$, which can also be written $(2x-1)^2$.

A common error with a perfect square is to just square the two bracketed terms. This error in understanding forgets the two linear terms that are created from multiplying two linear brackets.

The common error is to **WRONGLY SAY** that
$(x+3)^2 \equiv x^2 + 3^2 \equiv x^2 + 9$
But this is forgetting that $(x+3)^2$ does not mean $x^2 + 3^2$ but $(x+3)(x+3)$



Notice also that you always get double the constant in the bracket as your linear coefficient b. This fact is very important later for a technique called completing the square.

A **difference of squares** is where you have something like $(x+2)(x-2)$ to multiply out.
$(x+2)(x-2) \equiv x^2 - 2^2 \equiv x^2 - 4$
The two linear terms cancel one another and you end up with a difference (subtraction) of two square numbers x^2 and 2^2.
more complex example is (where $a \neq 1$)
$(2x-5)(2x+5) \equiv (2x)^2 - 5^2 \equiv\equiv 4x^2 - 25$

3) Factorise into 1 Linear Bracket

Earlier we learned that $3(x+2) \equiv 3x+6$

One method for calculating this was…

$3 \times x$ $\qquad$ 3×2

$(x$	$+\ 2)$	
$3x$	6	3

Because $3(x+2)$ and $3x+6$ are effectively the same thing, we should be able to start with $3x+6$ and put it into brackets to give $3(x+2)$

This process of putting things into brackets, is called factorising. In other words splitting the expression up into two factors that are multiplied together, just like 3 and 5 are factors of 15 because 3x5 = 15. So 3 and x+2 are factors of 3x+6 because $3(x+2) \equiv 3x+6$

The trick is to spot a factor that goes into both 3x and 6. I can see that 3 goes into both.

So we start with

$3x$	6	3

Knowing that 3 goes into both. We then have to find the other factor by seeing how many 3s go into each term.

$3x \div 3$ $\qquad$ $6 \div 3$

$(x$	$+\ 2)$	
$3x$	6	3

Hey presto, we have reversed the process of multiplying out a bracket, using the fact that 3 was a common factor of both terms…

$3x+6 \equiv 3(x+2)$

Let's try a quadratic (with c=0) with a common factor $6x^2+15x$

The factors 3 and x both go into each term, so we can use factor 3x.

$\frac{6x^2}{3x} = \frac{6}{3} \times \frac{x^2}{x}$ $\qquad$ $\frac{15x}{3x} = \frac{15}{3} \times \frac{x}{x}$

$(2x$	$+\ 5)$	
$6x^2$	$15x$	$3x$

Note that if we had started with a factor of either 3 or x, we would have ended up with a bracket that could still be factorised (by x or 3 respectively). You have to ensure you have found the highest common factor before doing your factorisation.

Hence $6x^2+15x \equiv 3x(2x+5)$

4) Factorise Quadratics (a=1)

We learned earlier that
$(x+3)(x+2) \equiv x^2+5x+6$

Similarly it must be true that
$x^2+5x+6 \equiv (x+3)(x+2)$
But we can't simply use a common factor, as there is no common factor that goes into x^2, $5x$, and 6. But we know from above that this expression can be factorised. Let's look at what we do know.
One thing we can say for sure, is that as we are looking for 2 linear brackets, they must both start with x because the only way to make x^2 here is with $x \times x$, so we'll need two brackets that look like $(x+?_1)(x+?_2)$

$(x$	$+\ ?_1)$	
x^2		$(x$
	6	$?_2)$

The next thing we know for sure is that the two constants (represented above with ?s) multiply to make 6. So the possibilities for them are…
1x6=6
2x3=6
-1x-6=6
-2x-3=6
Let's try multiplying out these four possibilities

$(x$	$+\ 1)$	
x^2	x	$(x$
$6x$	6	$6)$

$= x^2+7x+6$
The quadratic and constant terms are correct, but the linear term is wrong.

$(x$	$-\ 1)$	
x^2	$-x$	$(x$
$-6x$	6	$-6)$

$= x^2-7x+6$

Again, the quadratic and constant terms are correct, but the linear is wrong.

	$(x$	$-\ 2)$
$(x$	x^2	$-2x$
$-3)$	$-3x$	6

$$= x^2 - 5x + 6$$

Similarly, the linear term is not correct.

	$(x$	$+\ 2)$
$(x$	x^2	$2x$
$+3)$	$3x$	6

$$= x^2 + 5x + 6$$

This one has all three terms correct, so

$$x^2 + 5x + 6 \equiv (x+2)(x+3)$$

Now we deliberately chose all the pairs of constants in the linear brackets to make the correct constant for our quadratic expression, so the x^2 and constant term $+6$ worked out every time, but only when we took the correct pairing of possible factors $(x+2)$ and $(x+3)$ did we get the correct linear term.

For all quadratics where a=1, we will need $x \times x$, and so we need $(x +?_1)(x +?_2)$.

If we try all of the possible ?s who's product make the constant c, then one pairing should make the correct linear term.

We don't have to multiply out all the possibilities every time, we start with the one we think most likely (here as the linear term was positive, we wouldn't start with the negative pairs of factors). Hopefully we won't have to try too many before one works.

The huge time saving method of trying the most likely looking pair of constants first can be summarised in a table.

Let's look at $x^2 + bx + c = (x+m)(x+n)$

	b +^{ve}	b –^{ve}
c +^{ve}	m & n +^{ve}	m & n -^{ve}
c –^{ve}	m & n different signs (+^{ve} larger)	m & n different signs (-^{ve} larger)

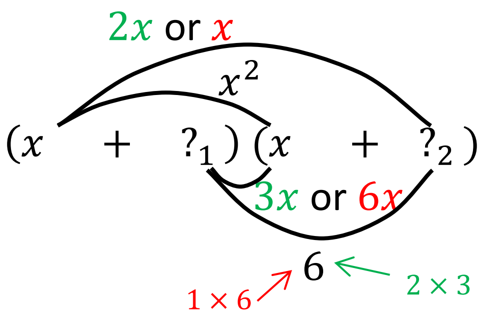
C can be made positive either with m & n are both positive or both negative. In the top row this shows the two possible outcomes of b, positive when both m & n are positive or negatve when they are both negative. Similarly of c is negative m and n must be different signs. The larger number of the two will dictate whether b is positive or negative.

For example if c is negative, then the two numbers m & n must be different signs. And when you add the two linear terms together, you end up with b which is positive. This means the positive number from m or n must be the bigger, it must be greater than the negative number to make their sum positive. So if m & n were either –3 and 7 or -7 and 3, because we know the positive number must be greater (to give a +ve b), then it must be 7 & -3.

Note: Use caution with this table when doing quadratics with $a \neq 1$ as sometimes the different 'a' can trump the effects in the above table!

Eventually we learn to do this without drawing out the grid, and we get quicker at deciding which pair of constant factors to try first.

It can be done more quickly, using a visual claw like method. We wouldn't even try the negative factor pairs, so we just need to try 1&6 and 2&3



The red one doesn't work (it makes linear term $7x$) and the green one does make $5x$ as needed, so the green constants of 2 and 3 are the ones we need.

From such visual methods we can eventually factorise without writing anything down.

We'll just try one more factorisation here....

$$x^2 - 2x - 15$$

	$(x$	$+\ ?_1)$
$(x$	x^2	
$?_2)$		-15

Now -15
= 1 x -15
= -1 x 15
= 3 x -5
= -3 x 5

Because b is negative I will need to try either 1 x -15 or 3 x -5. It looks like 1 x -15 will have too large a negative difference so I will try 3 x -5 first.

$(x \quad + \quad 3)$

x^2	$3x$	$(x$
$-5x$	-15	$-5)$

$$= x^2 - 2x - 15$$

This works, so
$x^2 - 2x - 15 \equiv (x+3)(x-5)$

Without drawing all the grids we will just multiply out the other 3 (mentally) to show that they have the same quadratic and constant terms (coefficients a and c) but a different linear term (coefficient b).

$$(x+3)(x-5) \equiv x^2 - 2x - 15$$
$$(x-3)(x+5) \equiv x^2 + 2x - 15$$
$$(x+1)(x-15) \equiv x^2 - 14x - 15$$
$$(x-1)(x+15) \equiv x^2 + 14x - 15$$

One also needs to be ready to factorised quadratics that are perfect squares, or difference of squares (b=0, c<0). The above methods still work for either of these types of quadratics, but understanding their special qualities makes factorising them easier and more fun!

Let's factorise difference of squares $x^2 - 49$
It is a difference (subtraction) of two squares x^2 and 7^2. Knowing that we need opposite signs of constant to cancel the linear terms (as the linear term here is $0x$) we can directly move to $(x+7)(x-7)$.

So $x^2 - 49 \equiv (x+7)(x-7)$

For a perfect square, let's take $x^2 + 6x + 9$, if we notice that c is a square that is half of b (ie $\sqrt{c} = \frac{1}{2}b$) then we can proceed directly to $(x+3)^2$ which is just a simpler way of writing $(x+3)(x+3)$.
Mathematicians almost always write their final answer in the most simplified form (eg writing $\frac{1}{3}$ rather than $\frac{4}{12}$ even though they mean the same thing). So we would write this as

$$x^2 + 6x + 9 \equiv (x+3)^2$$

5) Factorise Quadratics (a≠1)

Factorising quadratics with the quadratic co-efficient not 1, is somewhat more complicated. For a start, we get more options about the linear terms inside our brackets.

Let's look at $2x^2 + 5x + 3$

The quadratic term must be made from multiplying x and $2x$ so we will need
$(2x + ?_1)(x + ?_2)$

$(2x \quad + \quad ?_1)$

x^2		$(x$
	3	$+?_2)$

Now 3 can be made with 1x3 or -1x-3
Although we can ignore the negative values (see previous step), as we are trying to make linear term $5x$ which is positive and can't be made of two negative linear terms. But the 1 and the 3 will give different values depending on which way we put them, as one of them will be doubled by the $2x$.

$(2x \quad + \quad 1)$

x^2	x	$(x$
$6x$	3	$+3)$

$$= x^2 + 7x + 3$$

Which is not correct, let's try the other way.

$(2x \quad + \quad 3)$

x^2	$3x$	$(x$
$2x$	3	$+1)$

$$= x^2 + 5x + 3$$

So $(2x+3)(x+1) \equiv 2x^2 + 5x + 3$

And note that $(2x+1)(x+3)$ doesn't work!

Let's work with $2x^2 + 7x + 6$

Again the $2x^2$ must be $2x \times x$, but there are more options for the constant, 6.

$$6 = 1 \times 6 = -1 \times -6 = 2 \times 3 = -2 \times -3$$

If we tried all four both ways we'd have 8 possibilities to try, but again we can ignore the negatives as b is positive.

$$(2x+1)(x+6) \equiv 2x^2+13x+6$$

$$(2x+6)(x+1) \equiv 2x^2+8x+6$$

$$(2x+2)(x+3) \equiv 2x^2+8x+6$$

$$(2x+3)(x+2) \equiv 2x^2+7x+6$$

The last one is the one we need, so

$$2x^2+7x+6 \equiv (2x+3)(x+2)$$

Let's work with one last example where a is not a prime number. All the previous ones had a as a prime number, which meant there was only one way to get the quadratic term. But with, say, $4x^2$ you have options for the linear factors. Either $x \times 4x$ or $2x \times 2x$

Let's look at $4x^2+20x+9$
So $4x^2 \equiv x \times 4x \equiv 2x \times 2x$

And $9 = 1 \times 9 = -1 \times -9 = 3 \times 3 = -3 \times -3$

Again we can ignore all the negative pairs as b is positive. So we can try all the combinations of x & $4x$ and $2x$ & $2x$ along with the combinations of 1 & 9 and 3 & 3. This gives lots of possibilities (that have already been halved by ignoring the negative pairs of factors).

However we just have to work through the possibilities, trying to make intelligent guesses about which one to try first.

First guess: $(x+3)(4x+3) \equiv 4x^2+15x+9$

Second Guess: $(2x+1)(2x+9) \equiv 4x^2+20x+9$

There we go, we got it right second time.

So $4x^2+20x+9 \equiv (2x+1)(2x+9)$

6) Completing the Square

Look back at the end of step 2, about perfect squares. You notice that

$$(x+m)^2$$
$$\equiv x^2+mx+mx+m^2$$
$$\equiv x^2+2mx+m^2$$

In other words, the linear coefficient (b) is 2 lots of m (the constant in the bracket). So for any quadratic, we can halve b (the linear coefficient) and put this in a bracket that we then square.

Let's look at $x^2+6x+?$

If we halve b we get 3, then put make that the constant in a bracket and square it.

$$(x+3)^2 \equiv x^2+6x+?$$

So with this technique, we only need to get the ?s to match up which can always be done by adding or subtracting a constant on the outside of the bracket.

Let try with $x^2+6x+?$
So we try $(x+3)^2 \equiv x^2+6x+9$

It is very close but the constant c is wrong. It is 9 but it should be 7. So we just need to subtract 2 from both sides…

$$(x+3)^2 \equiv x^2+6x+9 \quad \textbf{(-2)}$$

$$(x+3)^2-2 \equiv x^2+6x+7$$

|So $x^2+6x+7 \equiv (x+3)^2-2$ in completed square form.

Now if b is an odd number, then when we halve it we get a fraction. The process is EXACTLY the same, AND we have to be skilled at manipulating fractions (read back over your fractions ladder if you are not confident about this!).

We'll complete the square for
$2x^2+5x+2$

So our "first guess" is $\left(x+\frac{5}{2}\right)^2$
Now $\left(x+\frac{5}{2}\right)^2 \equiv x^2+5x+\frac{25}{4}$
Then again we just need to adjust the constant. We need to get from $\frac{25}{4}$ to 2, which is $\frac{8}{4}$ so we need to subtract $\frac{17}{4}$

$$\left(x+\frac{5}{2}\right)^2-\frac{17}{4} \equiv x^2+5x+2$$

So $x^2+5x+2 \equiv \left(x+\frac{5}{2}\right)^2-\frac{17}{4}$ in completed square form,

Finally we need to look at quadratics where $a \neq 1$ like
$2x^2+12x+11$
Now $2x^2+12x \equiv 2(x^2+6x)$ so we will use "first guess" $2(x+3)^2$ which gives
$2(x+3)^2 \equiv 2(x^2+6x+9) \equiv 2x^2+12x+18$
And finally we can just adjust the constants by subtracting 7 from each side to give c=11.

$$2(x+3)^2 \equiv 2x^2+12x+18 \qquad (-7)$$
$$2(x+3)^2-7 \equiv 2x^2+12x+11$$

So $2x^2+12x+11 \equiv 2(x+3)^2-7$ in completed square form.

7) Simplify Quadratic Algebraic Fractions

Simplify $\dfrac{x^2+3x+2}{x^2-x-6}$

The trick to simplifying quadratic fractions (where they can be simplified) is in factorising them, and looking for common factors on the top and bottom that can be cancelled. So let's factorise the top and the bottom of our example.

$$\frac{(x+1)(x+2)}{(x-3)(x+2)}$$

Then separate out the common factors

$$\equiv \frac{(x+1)}{(x-3)} \times \frac{(x+2)}{(x+2)}$$

And cancel

$$\equiv \frac{x+1}{x-3}$$

And here's one where the quadratic terms have non-zero co-efficients.

$$\frac{2x^2+5x+3}{3x^2-x-4}$$

Factorise the top and bottom

$$\equiv \frac{(2x+3)(x+1)}{(3x-4)(x+1)}$$

Separate the common factors

$$\equiv \frac{(2x+3)}{(3x-4)} \times \frac{(x+1)}{(x+1)}$$

And cancel

$$\equiv \frac{2x+3}{3x-4}$$

8) + - x ÷ Algebraic Fractions

Multiplying and dividing quadratic fractions is usually easier than + & - them, so we'll start there.

$$\frac{x+2}{2(x-1)} \div \frac{x+2}{2(x+3)}$$

First we flip 2nd fraction & change it to a multiply

$$\frac{x+2}{2(x-1)} \times \frac{2(x+3)}{x+2}$$

Then times the tops and the bottoms out

$$\equiv \frac{(x+2)2(x+3)}{2(x-1)(x+2)}$$

And rearrange to make the common factors clearer

$$\equiv \frac{2(x+2)(x+3)}{2(x+2)(x-1)}$$

Then separate the common factors

$$\equiv \frac{2(x+2)}{2(x+2)} \times \frac{(x+3)}{(x-1)}$$

And finally cancel the common factors

$$\equiv \frac{x+3}{x-1}$$

Now let's look at a subtraction example (just like with constants the method for subtracting and adding fractions is very similar).

$$\frac{x}{x+1} - \frac{1}{(x+1)^2}$$

First we find equivalent fractions with common denominators; both bottoms will go into $(x+1)^2$

$$\equiv \frac{x}{x+1} \times \frac{x+1}{x+1} - \frac{1}{(x+1)^2}$$

$$\equiv \frac{x(x+1)}{(x+1)^2} - \frac{1}{(x+1)^2}$$

Then simplify and subtract

$$\equiv \frac{x^2+x}{(x+1)^2} - \frac{1}{(x+1)^2}$$

$$\equiv \frac{x^2+x-1}{(x+1)^2}$$

Before finishing let's just double check that the top doesn't factorise, in which case we might be able to simplify further and cancel some linear factors on top and bottom.

$(x+1)(x-1)$ is the only possibility, but this gives $(x+1)(x-1) \equiv x^2-1$ so x^2+x-1 doesn't factorise.

So $\frac{x}{x+1} - \frac{1}{(x+1)^2} \equiv \frac{x^2+x-1}{(x+1)^2}$