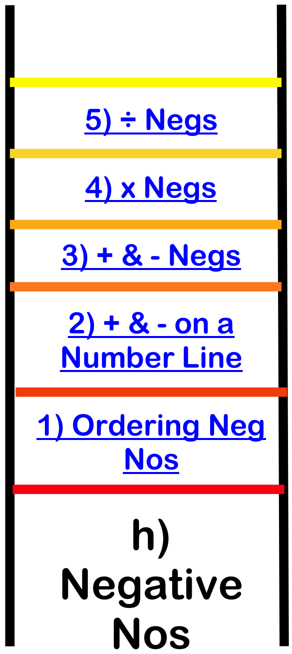
j) Understanding Negative Numbers

1 2 3 4 5



Step 1) Ordering Negative Numbers

For much of human history there was no number zero, no way of writing it down as a number when you have nothing. Now of course we have 0. Counting 1, 2, 3 happened for tens of thousands (possibly millions) of years before the idea of zero as a number. If there weren't nought, there weren't nought, no need for a concept, idea or certainly not a number to define it.

So eventually we learned that we could count 0, 1, 2, 3, and we could count this backwards, 3, 2, 1, 0… But what happens if we continue counting backwards after that.

How do you measure debt, I owe you money, so I have less than nothing. We will see more analogies later!

Let's look at what happens.

If we count down from 3, 2, 1, 0 and continue, we count to -1 (read minus one). This is like owing one instead of having one. Or if it was the height of a pile of earth, this object would be a 1m deep hole in the ground. Then we go -2 (read minus 2), so I owe 2, or the hole is 2m deep!

We now see that -2 is less than -1. If I owe 2 I have less than if I only owe 1. A 2m hole is less height above the ground than a 1m hole.

And so we continue and the number line continues backwards with minus numbers, like a mirror of the positive numbers -1, -2, -3, -4, -5…

The numbers -10 to 10 go in this order (smallest first)

-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10

Temperature in ^{0}C goes from -273^{0}C up past 0^{0}C (where ice melts), past 100^{0}C (where water boils) and on up to unlimited hotter and hotter temperatures.

Let's try and put in order the temperatures of some different days in a Scottish winte, starting with the coldest.

The days were 5^{0}C, -3^{0}C, 0^{0}C, -9^{0}C, 4^{0}C,& -7^{0}C.

If we highlight these temperatures on our -10 to 10 number line we can see their order very clearly.

-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10

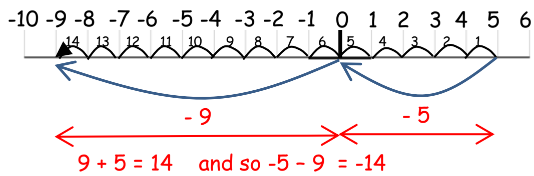
We can see that from coldest to hottest we get: -9^{0}C, -7^{0}C, -3^{0}C, 0^{0}C, 4^{0}C then 5^{0}C

The idea of a negative number, and how to order them, allows whole numbers (you could call the negative ones hole-whole numbers) to go on for ever in both directions, positive and negative.

So you can have a million or more, AND minus a million or less!

Step 2) + and – on a Number Line

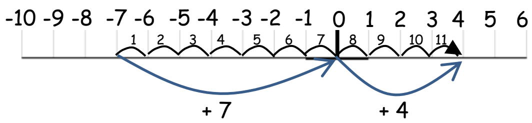
Let's look at the calculation 5 – 14. When we subtract we always move left on a number line, counting downwards. (The phrase counting down comes from viewing counting on a vertical number line). From 5, if we count down 5 we get to 0, but as we are subtracting 14 here we have to continue counting down (or left on the number line) into the negative numbers. You can see on this number line with the black arrows that counting back from 5 through 14 single number steps we end up at -9.



However we can use the fact that 5 -5 is 0 to calculate this quicker. 5 – 14 is the same as 5 – 5 –

9 (because 5 and 9 is 14, so taking both those numbers must be minusing 14). Now the 5 – 5 bit is 0, so we are left with -9. Have a look at the blue arrows and the red lines for a picture of this idea.

Now we'll look at the calculation -7 + 11. From the black arrow steps you can see we get to 4 if we add 11 single numbers to -7 (moving right on the number line to count "up") we get to 4 (we don't always say plus four, but we assume if we don't say either way we are talking about a positive number).



But 11 = 7 + 4

-7 + 7 + 4 just gives you 4 (as -7 + 7 is 0). You can see how this step links to basic addition facts.

It is worth repeating here that as adding and subtracting are opposites (or inverses), adding 3 and then adding 4 is the same as adding 7, and hence subtracting 3 and then subtracting 4 are the same as subtracting 7. This is how we can use number facts to quickly count up and down past zero.

Step 3) + & - Negative Number

Let us now look at what happens when we add or subtract negative numbers.

We will start with what we already know. If we add a positive number to a positive number, it gets bigger (further from zero in the upward direction).
so 5 + +3 = 5 + 3 = 8

If we take away a positive number that is the same as just taking away the number.

so 5 - +3 = 5 – 3 = 2

But what if we take away a minus number… surely to take away something negative we have to add something.

so 5 - -3 = 5 + 3 = 8

And finally what happens if we add a minus number. Even though I may say I am giving you something, the thing I am giving you is actually negative, so I am in fact taking something away from you.

so 5 + -3 = 5 - 3 = 2

These four combinations can be summed up as follows…

$$\left.\begin{matrix}a + +b \\ a - -b\end{matrix}\right\} a + b$$

$$\left.\begin{matrix}a + -b \\ a - +b\end{matrix}\right\} a - b$$

Or in this table…

	+	-
+	+	-
-	-	+

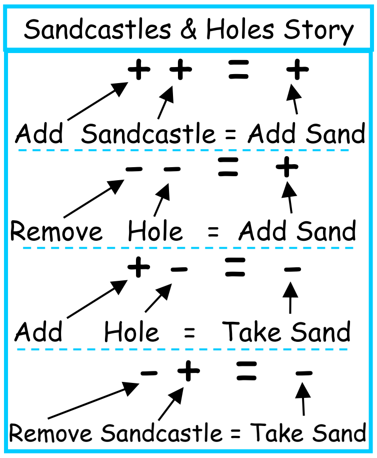
We could jump forwards now and say okay fine

++ = + (easy)
+- = - (hmmmm)
-+ = - (hmmmmmmm)
and -- = + (whaaaat!?!?!?)

The profound importance of these concepts is not one to be skipped over. Because times and dividing can be thought of as repeatedly adding or subtracting, the whole of the next two steps also uses these facts.

Here are three different ways of thinking about why these facts are true. The more you can understand why, the better you will be able to work with negative numbers.

These stories are the sand castle and holes story, the money and debt story and the bath story.



Money & Debt Story

+	+	=	+
Give	Money	=	Better Off
−	−	=	+
Remove	Debt	=	Better Off
+	−	=	−
Give	Debt	=	Worse Off
−	+	=	−
Remove	Money	=	Worse Off

The Bath Story (Hot & Cold Water)

+	+	=	+
Add	Hot Water	=	Hotter
−	−	=	+
Remove	Cold Water	=	Hotter
+	−	=	−
Add	Cold Water	=	Colder
−	+	=	−
Remove	Hot Water	=	Colder

Good & Bad Things

+	+	=	+
Add	Good Thing	=	Good
−	−	=	+
Take Away	Bad Thing	=	Good
+	−	=	−
Add	Bad Thing	=	Bad
−	+	=	−
Take Away	Good Thing	=	Bad

Can you invent your own story, your own way of thinking about negative numbers?

Step 4) x Negative Numbers

Let us try to multiply by some negative numbers.

let's look at 4 x -3. The starting point is actually +4 x +3 = 4 x 3 = 12. But what this actually means 4 lots of 3, or in other words repeatedly add 3 to itself 4 times. so 4 x 3 = 3 + 3 + 3 + 3 = 12

So 4 x -3 must mean add -3 together 4 times. Which is -3 + -3 + -3 + -3 = -3 -3 -3 -3 = -12

Next we can consider -4 x 3. How can you take minus 4 lots of a number (here the number is 3).

Well let's look at how we can take -1 lot of a number. From step 2 we could think about it as -- +_3 which is -3. Using our stories, what is minus one lot of 3? If you think of the debt story, minus one lot of having £3, would be owing £3. Or minus 1 lot of having a 3m metre high sand castle, would be having a three metre deep hole in the sand. Any which way, -1 lot of 3 is -3.

now -4 x 3 is 4 lots of -1 x 3 so it will be -3 -3 -3 -3 which is -12.

So -4 x -3 = -12.

Finally what if we have -4 x -3.

Again we'll start by considering -1 lot of -3. Using step 2 we can just call it - -3 which is +3. But let's look into our stories. What if I remove some cold water (- -), it must get hotter (+). What if I remove some debt from you (- -), you are actually better off (+). To remove a hole (- -) I have to add sand (+).

so -1 x -3 = +3

Now -4 x -3 is doing this 4 times, so it is +3 +3 + 3 + 3 which is 12.
Hence 4 x 3 = 12
-4 x -3 = 12
4 x -3 = -12
-4 x 3 = -12

In general this can be summed up as follows…

$$\left.\begin{matrix} + \times + \\ - \times - \end{matrix}\right\} + \qquad \left.\begin{matrix} + \times - \\ - \times + \end{matrix}\right\} -$$

Or again using the same table as for step 2.

		1st Number	
		+	−
2nd Number	+	+	−
	−	−	+
		Product	

From the sign of the 1st and 2nd numbers you multiply you can find the sign of their product.

Step 5) ÷ Negative Numbers

There are two ways to think about this. One is to consider dividing as the opposite (inverse) of multiplying. And we can also think of divide as a how many goes into question.

As 4 x 3 = 12

$\frac{+12}{+3} = +4$ and $\frac{+12}{+4} = +3$

So the same must be true for similar calculations with negatives.

As -4 x -3 = 12

$\frac{-12}{-3} = +4$ and $\frac{-12}{-4} = +3$

As 4 x -3 = -12

$\frac{-12}{+3} = -4$ and $\frac{-12}{+4} = -3$

And as -4 x 3 = -12

$\frac{-12}{+3} = -4$ and $\frac{-12}{-4} = +3$

We can see that the results are just like for timesing. The same signs multiplied make a positive, as they do divided. And different signs multiplied make a negative as they do divided.

$$\left.\frac{+}{+} \;\&\; \frac{-}{-}\right\} +$$

$$\left.\frac{-}{+} \;\&\; \frac{+}{-}\right\} -$$

But we can also look at this more directly, what does -12 ÷ 3 actually mean. It means divide -12 between 3 equal groups and how many in each group. It is -4. If we start with a negative quantity and divide it between an equal number of groups we will always get a negative number in each group.

If we use the question how many goes into for this question we can think of it in another way still.

Let's look again at -12 ÷ 3. It can be thought of as how many 3s go into -12. Well you can't get any positive number of 3s to make -12 as a positive multiplies by a positive will always be positive. 3 x 4 = +12 which is good, but the wrong side of zero! But 3 x -4 = -12 as it is just -3 -3 -3 -3 = -12.

Let's look at a dividing by a negative number… 15 ÷ -5. How many -5s go into 15, the answer is -3 as we need to do – (-5) – (-5) – (-5). We have subtracted the three -5s so it is -3 x -5.

Similarly for -21 ÷ -7. Hhow many -7s make -21. It is 3 as -7 -7 -7 = -21 you need three of them, so -21 ÷ -7 = 3.

Again the table can be a shorthand for this…

		Numerator	
		+	-
DEnominator	+	+	-
	-	-	+
		Gozzinta	