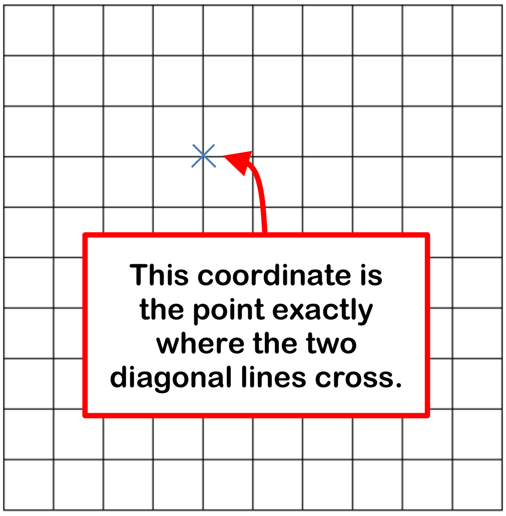
v) Understanding Graphs

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

v)
Graphs

Step 1) Positive Coordinates

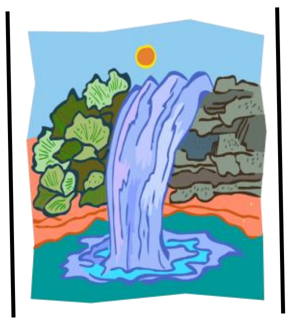
A coordinate is a point in space, usually marked on a flat grid on a piece of paper (or a screen) with a cross (or sometimes a dot). The grid has two dimensions. We can call the sideways or left—right dimension the horizontal dimension, and the up-down one the vertical dimension. The lines on the grid help us to trace the eye sideways along the horizontal dimension and up and down along the vertical dimension.



Here is a memory method for remembering the words horizontal (left-right) and vertical (up-down)…

Horizontal is like the horizon

Vertical, is like a waterfall

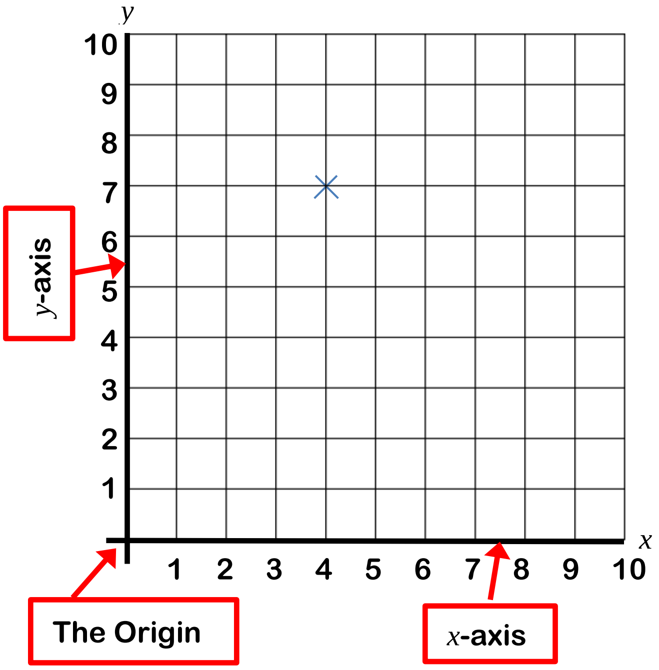


The “centre” point of the grid is what we call the origin. All coordinates are found by measuring from this point called the origin.

To talk about the position of the point on the grid, and exactly where it is relative to the origin, we draw two lines called axis.

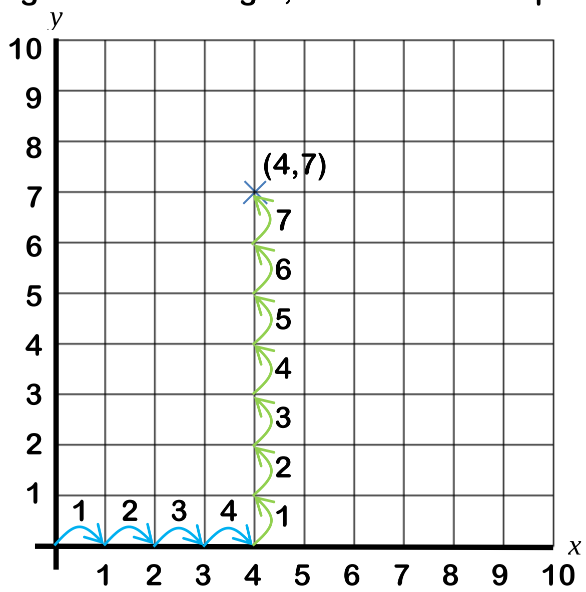
The horizontal axis (also called the *x*-axis) goes along the bottom line of the grid. We mark points 1 square away from the origin, 2 squares, 3 squares and so on, moving to the right away from the origin.

Then moving vertically upwards we mark the points 1 square, 2 squares etc away from the origin. This is the vertical axis (also known as the *y*-axis).



We find the coordinates from the origin, firstly by travelling sideways along the left-right dimension (labelled *x* on the axes), and then by travelling up the vertical dimension (labelled *y*).

The coordinate (point) on the previous grid is 4 right from the origin, and then and 7 up.



We call this coordinate (4,7). All coordinates can be represented in this way with two numbers in brackets separated by a comma. The first number tells you the horizontal movement from the origin (in the same direction as the *x*-axis). The second number tells you the vertical movement from the origin (in the same direction as the *y*-axis).

Some memorable ways to think of this are:

(horizontal from origin, vertical from origin)
(along the corridor, up the stairs)
(*x*-coordinate, *y*-coordinate)
(go in line with *x*-axis, go in line with *y*-axis)
(run along, rise up) or (run, rise)

or ultimately just (*x*, *y*)

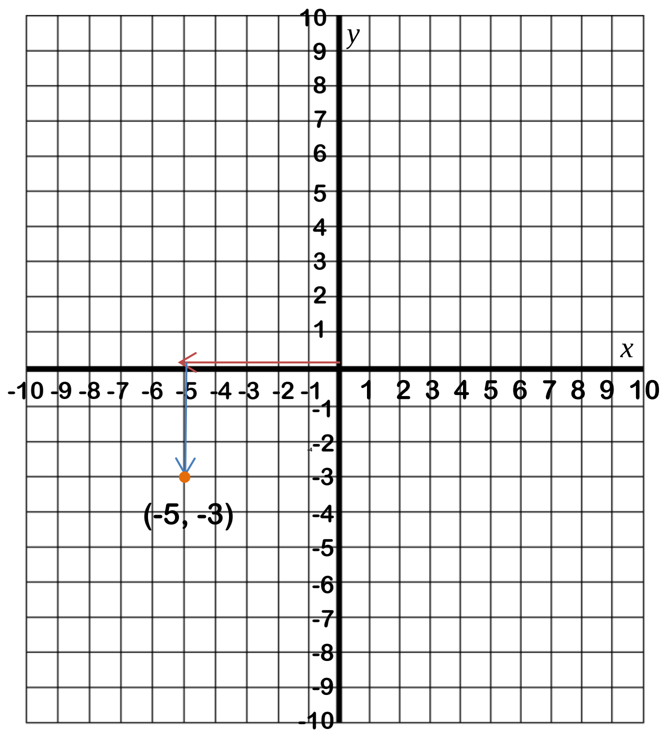
The origin as a coordinate is (0,0) as it is no distance from itself either horizontally or vertically.

Step 2) Negative Coordinates

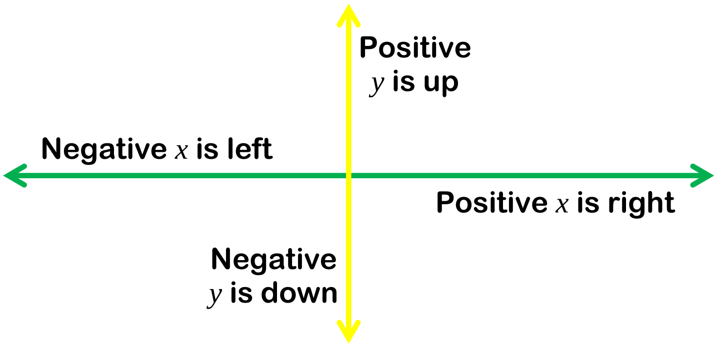
Coordinates are not limited to positive numbers. The axes can be extended past 0 in the negative direction. This means continuing the *x*-axis to the left after 3, 2, 1, 0 into -1, -2, -3 and so on. And the *y*-axis downward 3, 2, 1, 0 into -1, -2, -3 etc.

If the *x*-coordinate is negative it means move left. If the *y*-coordinate is negative it means move down.

For example, to find the coordinate (-5, -3) we go (from the origin) left 5 (horizontal -5) and down 3 (vertical -3)



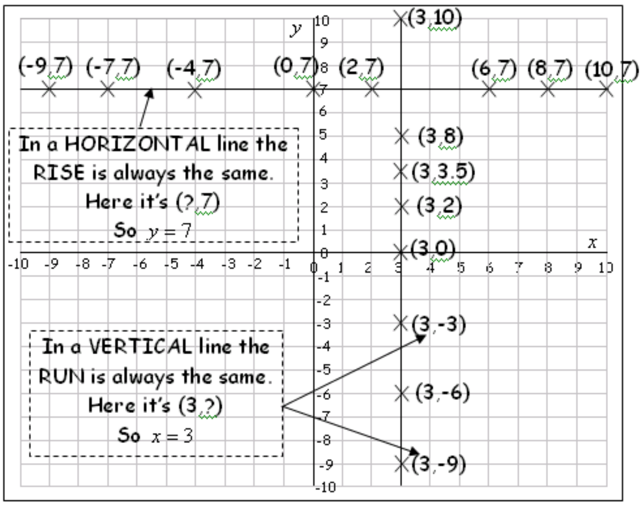
To summarise for coordinate (x, y):



[Step 3) Horizontal and Vertical Linear Graphs]

Horizontal and vertical lines have a very special property which links to their x and y coordinates.

On the axes below let's label several points on the horizontal line and several points on the vertical line.



Why is the y-coordinate always the same for coordinates on a horizontal line? Well a horizontal line is parallel to the x-axis, which means every point on it is the same vertical distance from the x-axis, which means the y-coordinates will all be the same.

Similarly any vertical line is parallel to the y-axis, so all points on it are the same distance horizontally from the y-axis and so it's x-coordinates will all be the same.

Hence a good name for the horizontal line that goes through
(2,9)
(9,9)
(-5,9)
(7,9)
(0,9)
(-17,9)

is simply $y = 9$, as the y-bit is always 9.

and a good name for the vertical line that goes through

(5,-10)
(5, 4)
(5,0)
and (5, -2)
is $x = 5$, as all the x-bits are 5.

In the same way the y-axis, which is vertical, is called $x = 0$, it goes through (0,-2) (0,5) (0,9) etc.
And the x-axis, which is horizontal, is called $y = 0$. it goes through (-2,0) (5, 0) (9,0) etc.

[Step 4) Plotting Diagonal Linear Graphs]

All the lines we will deal with in the next few steps will be straight. If you remember we name expressions where the largest term is a quadratic (like x^2, $7x^2$) quadratic expressions. Ones where the largest term is linear (like $3x$, $9x$, $-2x$) we call linear expressions. You can think of the name as line-ar (say line-eee-ah), meaning more liney! Any linear graph will always be a straight line so linear graphs really are very liney. (We'll see some of the amazing curved lines you get with higher powers of x in later steps.

Let's plot the graph of $y = 2x + 1$ for x from 0 to 4.

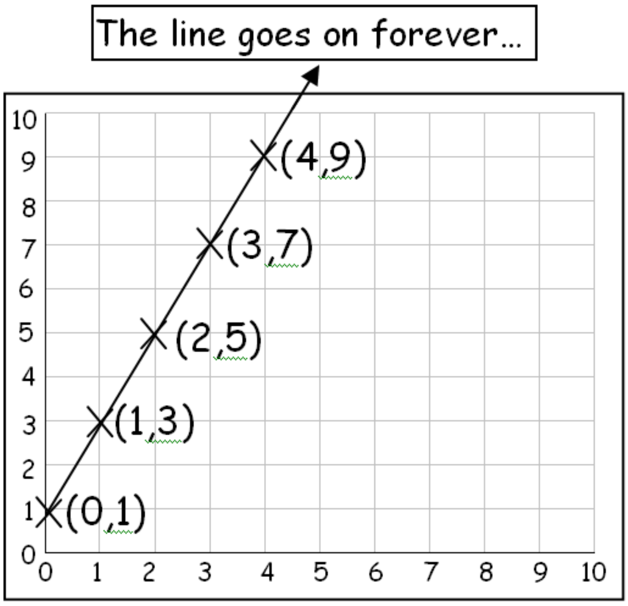
The tells you that y is found by doing some things to x (here first doubling it, then adding 1). There is in other words a relationship between the value of x and the value of y, if you know one part of the coordinate (x), you can you can find the other part of the coordinate (y).

We can think of our coordinates (x, y) here as $(x, 2x + 1)$. In other words, to find the you y-values, we have to substitute the x-values into $2x + 1$. Let's do that in a table.

x	0	1	2	3	4
Calculation to find y	2x0 +1	2x1 +1	2x2 +1	2x3 +1	2x4 +1
y	1	3	5	7	9
Coordinates	(0,1)	(1,3)	(2,5)	(3,7)	(4,9)

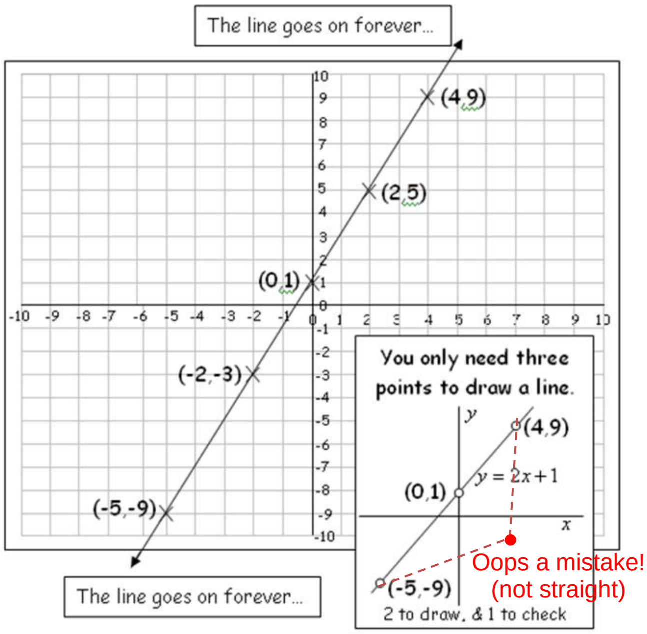
Let's look in detail at the coordinate where $x = 2$. Because $x = 2$ we know our coordinate will look like (2, y). We can find that y-value using the formula $y = 2x + 1$. Multiply x (which here has value 2) by 2 and then add 1. So $y = 2 \times 2 + 1 = 5$. So the coordinate is (2,5).

In the table we have done this for all the x-coordinates we were asked to work with. Now we can plot the coordinates on axes, and join up the points with a straight line, extending it in both directions past the last points.



Hooray, we have drawn our first diagonal graph. Let's try the same graph with negative values of x from -5 to 4.

x	-5	-2	0	2	4
Calc.	2x-5 +1	2x-2 +1	2x0 +1	2x2 +1	2x4 +1
y	-9	-3	1	5	9
Coords	(-5,-9)	(-2,-3)	(0,1)	(2,5)	(4,9)



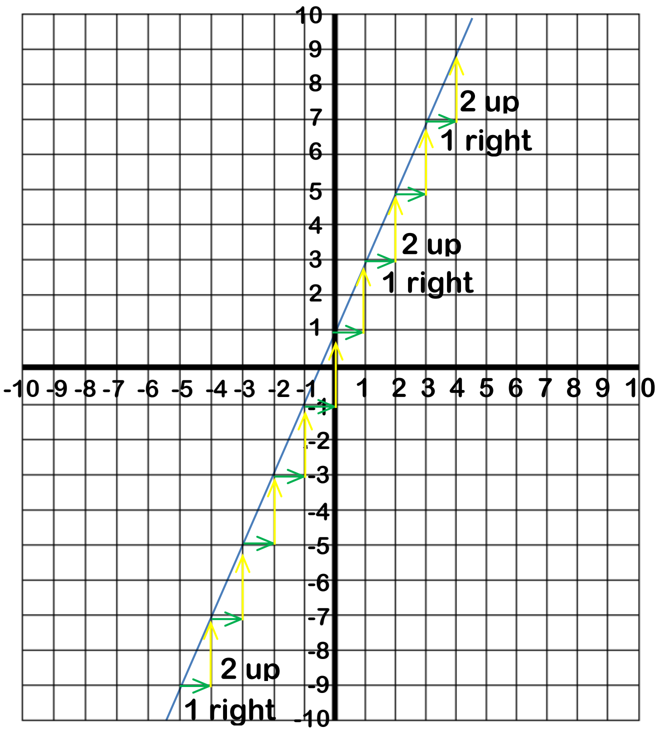
Here we calculated 5 different coordinates and plotted them. You can draw a diagonal line with just two points, but if you have made a mistake on one of them then your line will be waaaaay off! So the quickest safe way is to use three coordinates and if you make a mistake on one of them you'll see immediately, as the line won't be straight.

Step 5) Measuring Gradient m

Gradient is a word which describes how steep a line is. It is a number and the bigger the number the steeper the line.

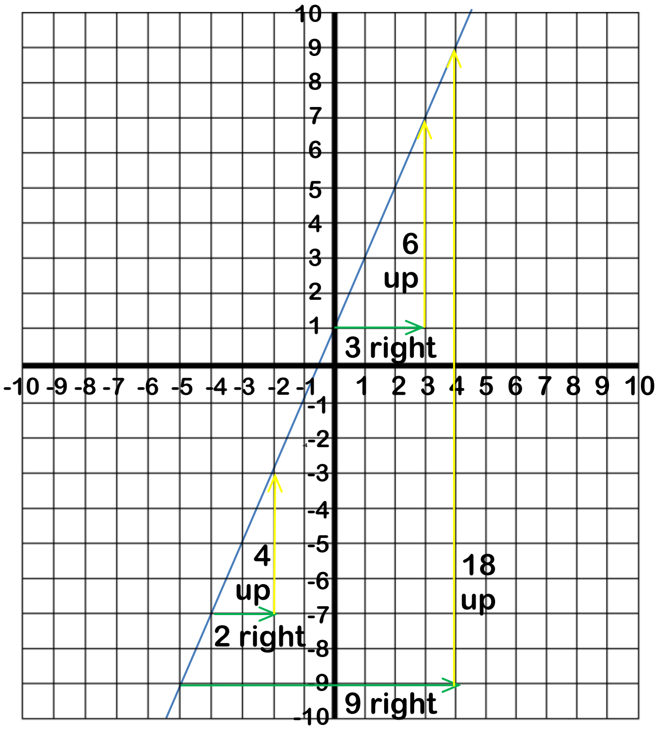
One way to think about it is how far you go up, for every 1 you move right.

So on the graph we just drew, which was represented by equation $y = 2x + 1$ we can step across 1 (here with the green arrows) and see how far we need to step up (here the yellow arrows).



Wherever we do this on the graph of $y = 2x + 1$ we have step up 2 for every 1 we step right. The gradient of this graph then is 2.

We can find the gradient not just by drawing little triangles 1 wide, any size of triangle will work. We can simply divide the step up by the step right. A memorable wording of this is rise over run.



We've drawn 3 triangles here, we can see that if you go 2 right, you have to go 4 up to reach the graph. For 3 right you must go 6 up. And finally for 9 right you have to go 18 up.

Lets try dividing the amount up by the amount right...

$$\frac{4}{2} = \frac{6}{3} = \frac{18}{9} = 2$$

They all equal 2. This is not magic. You can imagine 2 of our little 1 right, 2 up triangles, making one large 2 right 4 up triangle. Or 3 of them making the 3 right 6 up triangle.

The most advanced way of writing this is

gradient = $\frac{dy}{dx}$

Which means the difference in y divided by the difference in x. This might be confusing as it looks like it should mean d times y, divided by d times x, and where does d come from. This is not like usual algebra, it is a very special symbol meaning the difference in the variable (x or y here). But let's look at our three triangles...

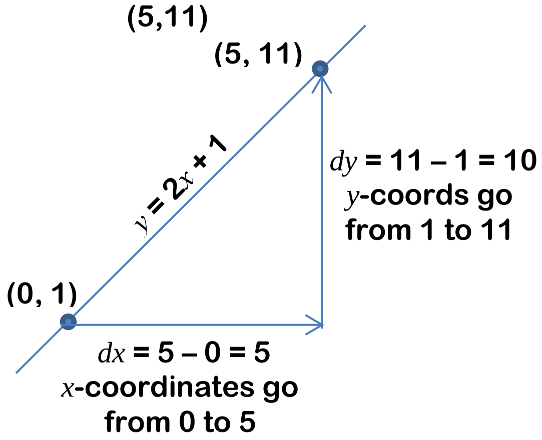
2 right, 4 up, $dx=2$, $dy = 4$, so $\frac{dy}{dx} = \frac{4}{2} = 2$

3 right, 6 up, $dx=3$, $dy = 6$, so $\frac{dy}{dx} = \frac{6}{3} = 2$

9 right, 18 up, $dx=9$, $dy = 18$, so $\frac{dy}{dx} = \frac{18}{9} = 2$

We don't actually need to do lots of different triangles, one will do. We don't even need a grid to count, the difference between the x & y coordinates will be enough.

Find the gradient of this graph through (0,1) (5,11)



We can make a formula for the difference between the y-coordinates and the difference between the x-coordinates.

For two points (x_1, y_1) & (x_2, y_2)

The gradient m,

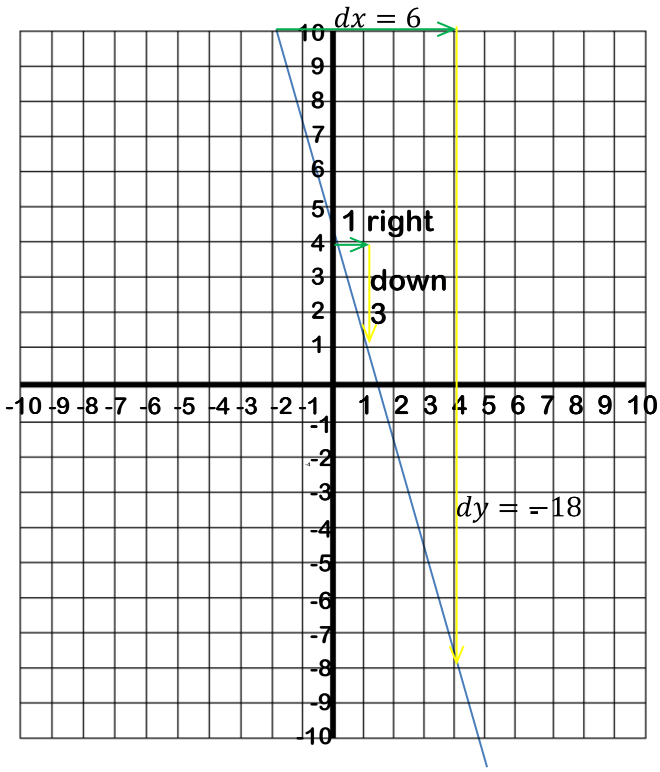
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(change in y, over change in x)

$$m = \frac{11-1}{5-0}$$

$$m = \frac{11 - 1}{5 - 0}$$

So $m = 2$



Because this line slopes downwards, when you step right one you have to step down 3 (not up 3. This means it has gradient -3.

To use the formula with our larger triangle, $dx = 6 \,\&\, dy = -18$

so $\frac{dy}{dx} = \frac{-18}{6} = -3$

which fits with our 3 steps down (vertical -3) for each step right (+1 horizontally).

6) Plotting Linear Graphs with y = mx +c

The general form of a linear graph is represented by the equation $y = mx + c$. As we saw in the last step, m represents the gradient, in other words how steep the line is.

For example $y = 2x + 1$ where $m = 2$ and $c = 1$
and $y = 2x - 3$ where $m = 2$ and $c = -3$

As we have already looked a bit at gradient in step 5, let's look at c.

In the two above equations, we are keeping m the same (it is 2 in both). This means the lines will have the same steepness, in other words they will be parallel.

Let's take a look at several more like this, where m is 2, but where c changes. We will particularly look at the points on the y –axis, in other words the point where $x = 0$ (see step 3 on horizontal and vertical lines).

Equation	x	y-calc	y	Coord
$y = 2x + 5$	0	$2 \times 0 + 5$	5	$(0, 5)$
$y = 2x + 4$	0	$2 \times 0 + 4$	4	$(0, 4)$
$y = 2x + 3$	0	$2 \times 0 + 3$	3	$(0, 3)$
$y = 2x + 2$	0	$2 \times 0 + 2$	2	$(0, 2)$
$y = 2x + 1$	0	$2 \times 0 + 1$	1	$(0, 1)$
$y = 2x$	0	2×0	0	$(0, 0)$
$y = 2x - 1$	0	$2 \times 0 - 1$	-1	$(0, -1)$
$y = 2x - 2$	0	$2 \times 0 - 2$	-2	$(0, -2)$
$y = 2x - 3$	0	$2 \times 0 - 3$	-3	$(0, -3)$
$y = 2x - 4$	0	$2 \times 0 - 4$	-4	$(0, -4)$
$y = 2x - 5$	0	$2 \times 0 - 5$	-5	$(0, -5)$

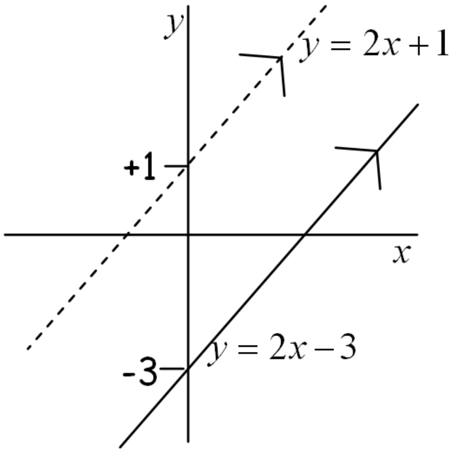
What we can see from this table, is that you always end up with a coordinate (0,c). In others words the y –coordinate always ends up being whatever c is. This is because x is 0, so whatever m is, 0 times m will still be 0, leaving you just with the value of c. Another way of writing this is to say that the point on the y-axis will be…

$$(0, 0 \times m + c)$$
$$= (0, 0 + c)$$
$$= (0, c)$$

The great thing about this is that you can use it to place your line. m tells you its steepness, in the case of both $y = 2x + 1$ and $y = 2x - 3$ this gradient (or steepness is 2), so the line goes right 1 and up 2. And c tells you where it crosses the y-axis. We call this point (0, c) the y-intercept (intercept just means cross, so it is where the line crosses the y-axis).

So for $y = 2x + 1$ the y-intercept is (0, 1)
and for $y = 2x - 3$ the y-intercept is (0,-3)

Now we can sketch these two parallel lines.

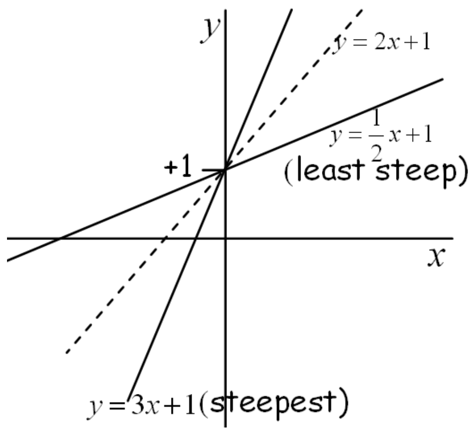


Playing around with m now, we can compare lines with different steepness, keeping c the same.

$y = 2x + 1$ where $m = 2$ and $c = 1$
$y = ½x + 1$ where $m = ½$ and $c = 1$
$y = 3x + 1$ where $m = 3$ and $c = 1$

y -intercepts are all 1 so (0, 1)

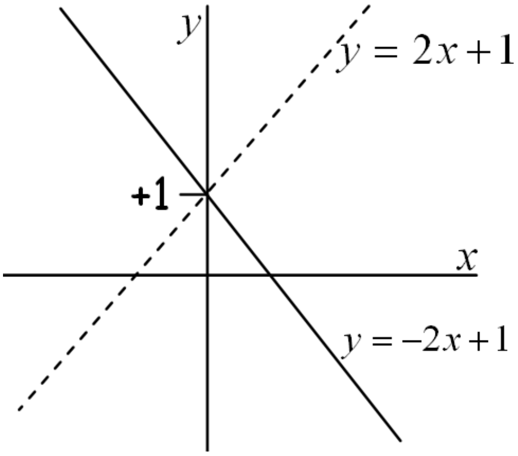
$m = 3$ is steepest,
$m = ½$ is least steep



It is also important to remember that a negative m, means a downhill slope. So m of 2 and -2, will have the same steepness, with 2 uphill, and -2 downhill.

$y = 2x + 1$ where $m = 2$ and $c = 1$
$y = -2x + 1$ where $m = -2$ and $c = 1$

y-intercepts are both 1 so (0, 1)



It can take a lot of practice to get your head around these two concepts. Two things that can help are sketching lots of lines, being given a first line and sketching in more based on whether they or steeper or less steep (as m changes) or move up and down (as c changes).

We'll take one last look at why adding one to c, will make a parallel line 1 higher up in the y-direction.

x	-3	-2	-1	0	1	2	3	4
$y = 2x + 1$	-5	-3	-1	1	3	5	7	9
$y = 2x + 2$	-4	-2	0	2	4	6	8	10

The y-values of $y = 2x + 2$, are just 1 more than the y-values of $y = 2x + 1$, so the whole line moves up 1 (in the positive *y*-direction).

One last way we can think about why m is the steepness is from thinking about our first explanation of gradient, after a step 1 to the right from the line, it is how far we have to move vertically back up (or down if it's a negative gradient) to the line.

Let's look at $y = 3x$ and then $y = mx$

x	1	2	3	4	5
$y = 3x$	3	6	9	12	15
Step up	+3	+3	+3	+3	

So every time we increase x by 1, we increase the value of $3x$ (and hence y) by 3. This makes sense as $3x$ with values of x as 1, 2, 3, 4 basically gives us the 3 times table which goes up in 3s. So for any line with $3x$ as its linear term (whatever the constant c is) you will step up in 3s each time you step right 1, and hence the gradient will be 3.

x	1	2	3	4	5
$y = mx$	m	2m	3m	4m	5m
Step up	+m	+m	+m	+m	

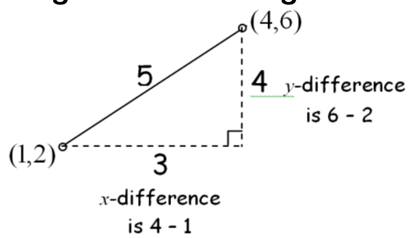
Similar to with $3x$, with mx every time we increase x by 1, we increase mx by m. Again we are looking at the m times table. We are going up by m for each step 1 to the right, so m will always give us the gradient.

Step 7) 2D Midpoints & Distances

Let's try and find the distance between the points

$(1,2)$ & $(4,6)$

This might seem very complex, and of course we could make a scale drawing and measure the distance. But if we put in the dx and dy lines, forming a right angled triangle, we can use Pythagoras on the length of dx and dy.



So dx and dy and the line we want to find the length of have given us a right angled triangle with the length we don't know being the hypotenuse. We can now apply Pythagoras…

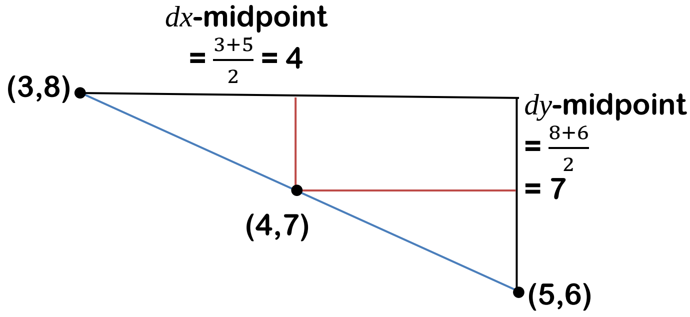
$$\sqrt{(4-1)^2 + (6-2)^2}$$
$$= \sqrt{3^2 + 4^2}$$
$$= \sqrt{9+16}$$
$$= \sqrt{25}$$
$$= 5$$

So the distance between $(1,2)$ & $(4,6)$ is exactly 5.

We can turn this into a formula using (x_1, y_1) & (x_2, y_2) as out points.

Distance = $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

Midpoints can be found using a similar method. We use half way along the x-difference (dx) for the x-coordinate, and half way along the y-difference (dy) for the y-coordinate. Let's find the midpoint of (3, 8) & (5, 6).



We can use this method to make a formula.

For two points (x_1, y_1) & (x_2, y_2)

The midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Average of x-coords

Average of y-coords

Using the formula with our example we get

$$\left(\frac{3+5}{2}, \frac{8+6}{2}\right)$$
$$= \left(\frac{8}{2}, \frac{14}{2}\right)$$
$$= (4,7)$$

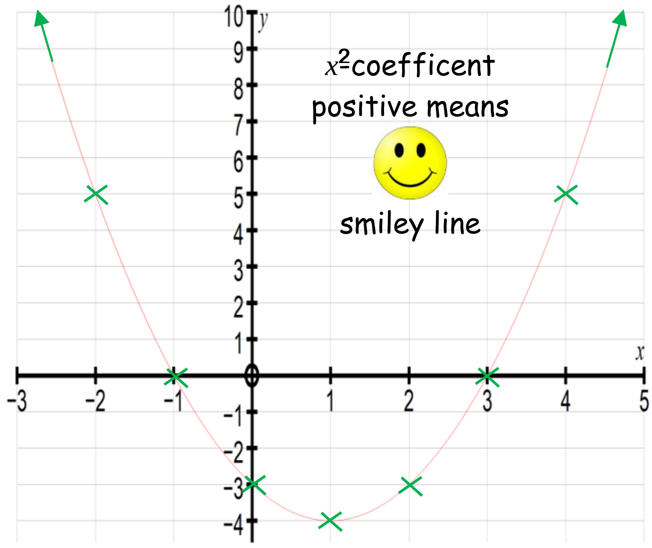
Step 8) Plotting Quadratics

Let's plot the graph of $y = x^2 - 2x - 3$ from $x = -2$ to $x = 4$.

Just like with a linear graph we can substitute some x-values into the equation to find the corresponding y-coordinates.

x	-2	-1	0	1	2	3	4
Terms	4 +4 -3	1 +2 -3	0 -0 -3	1 -2 -3	4 -4 -3	9 -6 -3	16 -8 -3
y	5	0	-3	-4	-3	0	5
Coords	(-2,5)	(-1,0)	(0,-3)	(1,-4)	(2,-3)	(3,0)	(4,5)

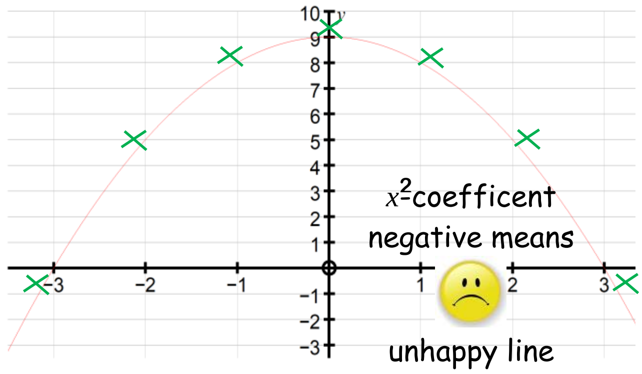
Then we can plot these coordinates on the axes, and join the points up with a line. The green arrows remind us that the line continues past the values we have decided to plot.



As you can see it doesn't make a straight line like a linear graph, it makes a curved line. This shape is called a parabola, and people sometimes use the image of a bola hat to remember the shape and its name. Though I prefer thinking of it as a quadratic smile ☺.

Negative quadratic coefficients still make a parabola shape, but they also look different. Let's try $y = 9 - x^2$

x	-3	-2	-1	0	1	2	3
Terms	9 - 9	9 - 4	9 - 1	9 - 0	9 - 1	9 - 4	9 - 9
y	0	5	8	9	8	5	0
Coords	(-3,0)	(-2,5)	(-1,8)	(0,9)	(1,8)	(2,5)	(3,0)



The shape of this one is like an upside down version of the previous graph. In other words it is a frown. In short any positive x^2 will make a smile and any negative x^2 will make a frown shaped curve (called a parabola).

Step 9) Sketching Quadratics (a=1)

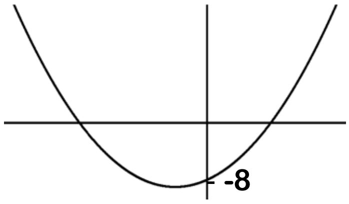
Plotting a quadratic means working out lots of points, but we can sketch the overall shape of a quadratic just using the points where it crosses the axes.

We call the point where the line crosses the y-axis the y-intercept.

And the point(s) where the line crosses the x-axis we call the root(s).

We already saw in the last step that for a positive co-efficient quadratic, you get a smiley parabola, and for a negative co-efficient quadratic you get a frowning parabola. Let's look at the y-intercept.

We'll work with $y = x^2 + 2x - 8$.
The y-axis is the line $x = 0$ so we are looking for a point $(0, y)$. Let's substitute in $x = 0$ to find the y-coordinate $y = 0^2 + 2\times0 - 8 = -8$, so the y-intercept is $(0, -8)$.



Next let's find the roots where the line crosses the x-axis. The x-axis is the line $y=0$, we substitute this value into $y = x^2 + 2x - 8$

$$0 = x^2 + 2x - 8$$

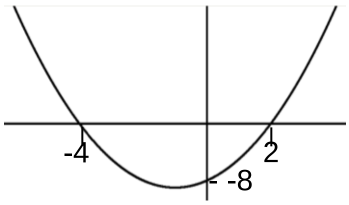
This is a quadratic equation we can solve using substitution.

$$0 = (x + 4)(x - 2)$$

so $x + 4 = 0$ or $x - 2 = 0$

so $x = -4$ or $x = 2$

So our roots (where the line crosses the x-axis) are (-4, 0_ and (2, 0). We simply mark these on our sketch along with the y-intercept.



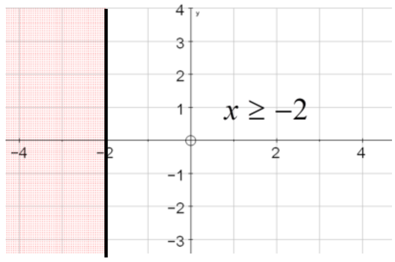
No other equation has these same three axis crossing points, so this sketch is unique for this one equation.

This sketching of eqautions is quicker than a full plotting, and can help us look at the gradient (steepness) of the line at different points, or where it might be flat (gradient = 0) or where two lines might cross one-another.

Step 10) Regions with Inequalities

Whenever we have an inequality, we get regions on the graph. An inequality tells us something is either greater than or less than, which means it must fall on one side of the equation's line. Any graph line creates two regions, one each side of the line, and with multiple graphs we can make all sorts of interestingly shaped regions.

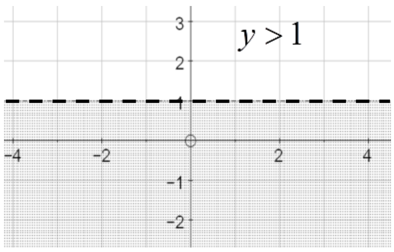
Let's look first at a region created by a vertical line $x \geq 2$. Now the two regions $x \geq 2$ and $x \leq 2$ must be separated by the line $x = 2$. So we plot the line $x = 2$.



Now when x gets bigger we move to the right so $x \geq 2$ must be the region on the right hand side of $x = 2$. Here I have labelled the region on the right $x \geq 2$ and shaded out the left hand side out in pink.

You can also shade in the region you want, but shading out the region you don't want has advantages with regions created by multiple graph lines.

Now we'll look at $y > 2$. First we plot $y = 2$. Then we have to work out whether our region is above or below it. y get's bigger when you move up, and this says y must be greater than, se we label the region above the line $y > 2$ and shade out (grey here) the region below the line.

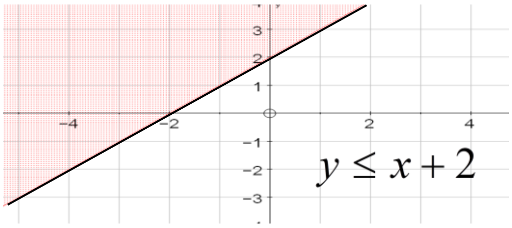


Because this inequality is strictly greater than (>) rather than greater than or equal to (≥) we use a dashed line to give a sense of the line not being included in the region.

Dashed line for $>$ & $<$

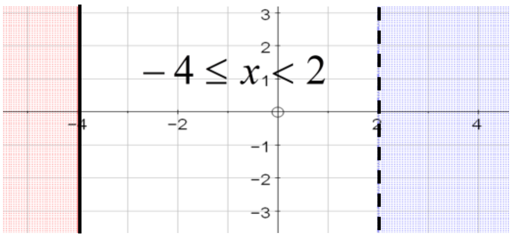
Solid line for $\geq$ & $\leq$

Now let's look at a region created by a diagonal inequality $y \leq x + 2$. We will use a solid line as it is $\leq$. We are looking for less than, and when y gets less we move down, so we want the region below the line. We label it $y \leq x + 2$ and shade out the region we don't want (here in pink).

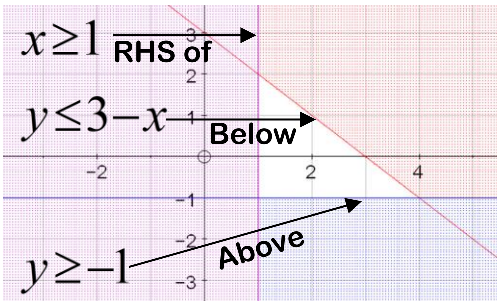


Let's look at a region created by two inequalities $-4 \leq x < 2$. We have to plot $x = -4$ with a solid line (as it's $\leq$) and $x = 2$ with a dashed line (as it is <). We shade out the region on the right (here blue) of the dashed line as x is is less than (which means left of), and we shade out the region on the left of the solid line as x is greater than, which means right of (here pink).

We are left with the unshaded region that fits both the inequalities.



Finally lets look at an enclosed shape form by three different inequalities, $x \geq 1$, $y \leq 3 - x$ & $y \geq -1$. They will all be solid lines as all $\leq$ or $\geq$. We plot all the lines and then for $x \geq 1$, we want the region on the right of the line (shade out left in lavender), f0r $y \leq 3 - x$ we want the region on under the line (shade out above in pink) & $y \geq -1$ we want the region above the line (shade out below in blue).



We get the region (a triangle) we want left unshaded in between the lines!

In summary..

$x <$ or $x \leq$ means the region left of

$x >$ or $x \geq$ means the region right of

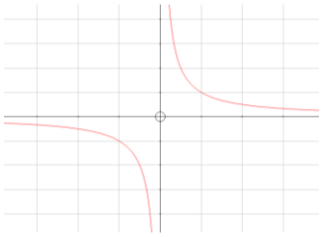
$y <$ or $y \leq$ means the region below

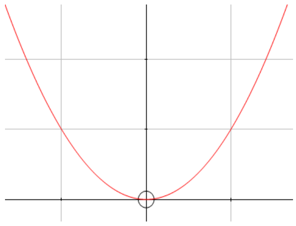
$y >$ or $y \geq$ means the region above

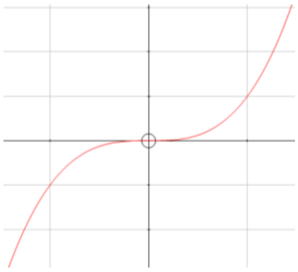
Step 11) Graph Shape Types

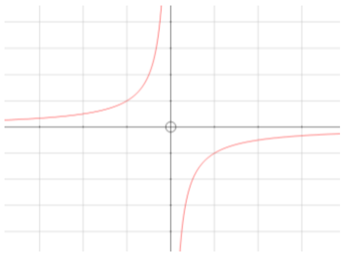
Let's look at a selection of graphs and try to match them with their shapes.

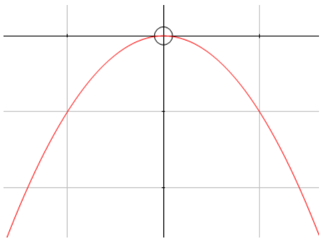
Two quadratics $y = 3x^2$, $y = -2x^2$
two cubics $y = x^3$, $y = -5x^3$,
and two reciprocals $y = \frac{7}{x}$, $y = \frac{-3}{x}$

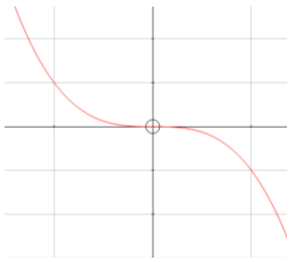






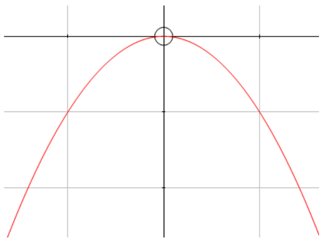


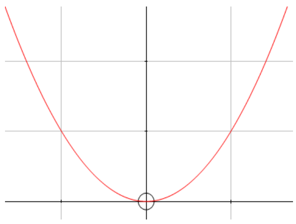




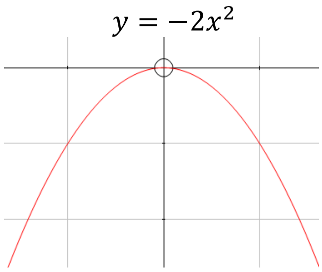
Firstly let's group them into the two pairs of similarly shaped graphs.

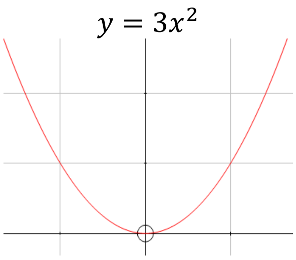
First the two smile/frown (or bola hat) graphs shaped





We know from step 8 that smiley and frown shaped graphs are called parabolas and are quadratic graphs. So our two options are $y = 3x^2$, $y = -2x^2$. We also learned there that $y = 3x^2$ must be a smile shaped parabola as 3 is positive, and $y = -2x^2$ must be a frown shaped parabola as -2 is negative.





We can look at this in more depth by seeing what happens as x grows. As x gets very large, whether x is positive or negative, we end up with a large positive value of x^2 (squaring a negative, is a negative times a negative which always makes a positive).

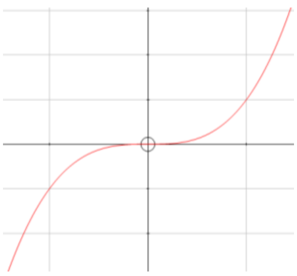
x grows large +ve $\longrightarrow$ large +ve x^2
x grows large -ve $\longrightarrow$ large +ve x^2

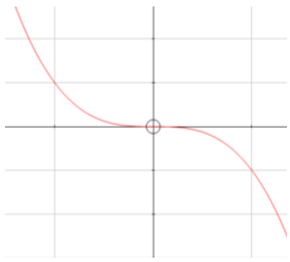
So either way we have +ve x^2

Then multiply +ve x^2 by a +ve coefficient gives a +ve y-value.
Hence the positive coefficient of x^2 will curve up towards large positive y-values at both extremes as x gets very large, giving a smile shape.

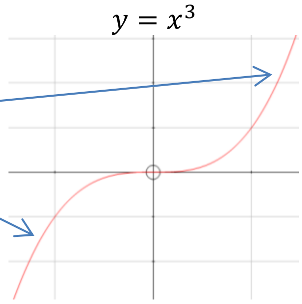
Similarly if we multiply +ve x^2 by a -ve coefficient gives a -ve y-value.Hence the negative coefficient of x^2 will curve down towards large negative y-values as x gets very large.

Next we'll look at the two S-shaped curves. They look like the parabola, but one side of the curve has been reflected in the x-axis.



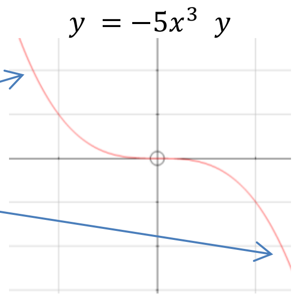


This is because they are x^3 curves. When you square a negative you get a positive, but when you cube a negative you get a negative. So it's a curve shaped like x^2 at the positive x end, and the reflection of that shape in the x-axis at the at the negative end.



So x^3 will curve up as x gets very +ve large, and curves down as x gets very –ve large.

If you multiply all this by a negative coefficient

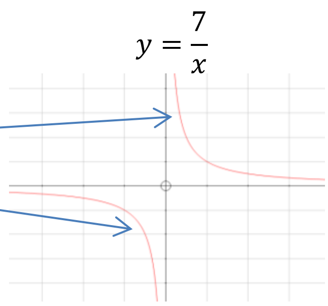


the curve slopes up as x gets very –ve large
and curves down as x gets very +ve large.
.

This leaves us with two interestingly shaped graphs, which relate to reciprocal graphs like $y = \frac{1}{x}$ or more generally $y = \frac{a}{x}$

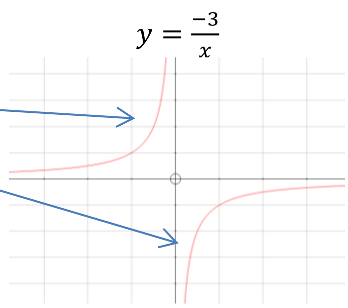
Let's look at the two options.

With $y = \frac{7}{x}$



x +ve $\longrightarrow$ y +ve
& x -ve $\longrightarrow$ y -ve

and with $y = \frac{-3}{x}$



x -ve $\longrightarrow$ y +ve
& x +ve $\longrightarrow$ y -ve

It is also interesting to see where the shape comes from.
as x gets close to zero, $\frac{1}{x}$ gets very large
as x very large gets, $\frac{1}{x}$ gets close to zero

So you have the curve in one quadrant getting close to the axis at extreme ends but never touching it. And you get the same thing at the opposite quadrant, as x –ve flips it the other side of the x-axis.

Step 12) Perpendicular Lines

Perpendicular lines have a relationship between their gradients.

$y = 2x + 7$ & $y = -\frac{1}{2}x - 3$ are perpendicular

$y = -3x + 2$ & $y = \frac{1}{3}x + 4$ are perpendicular

$y = -\frac{1}{9}x + 1$ & $y = 9x - 5$ are perpendicular

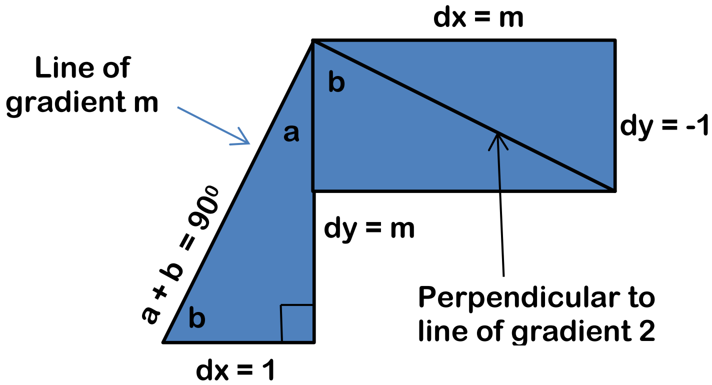
Try plotting them to see that they are perpendicular. Can you spot the relationship between the gradients?

The relationship is that the product of the gradients is -1.

So all lines perpendicular to the line $y = 3x - 1$ will be of form $y = mx + c$
3m = -1
so m = -⅓

So the form is $y = -\frac{1}{3}x + c$ (c can be anything)

The reason can be understood looking at two right angled triangles making up the dx and the dy for a line.

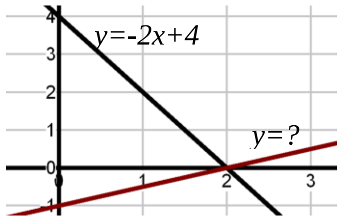


The original line has gradient a, here we have made a triangle with dx = 1, and dy = a. Moving the right angled triangle around we can create a second line perpendicular to it, which has dx = m and dy = -1.

So the second line has gradient $\frac{dy}{dx} = \frac{-1}{a}$

Multiplying the two gradients we get $\frac{-1}{a} \times a = -1$. So the product of the gradients will always be -1/

Let's find the eq$^{\underline{n}}$ of the red line with y-intercept (0,-1) perpendicular to $y = -2x + 4$.



Let's call it $y = mx + c$

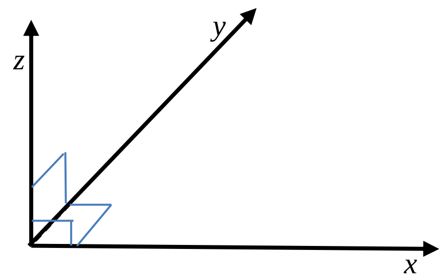
y-intercept (0,-1) tells us that c is -1

And because it is perpendicular to $y = -2x + 4$.
so -2m = -1
m = ½

$y = ½x - 1$

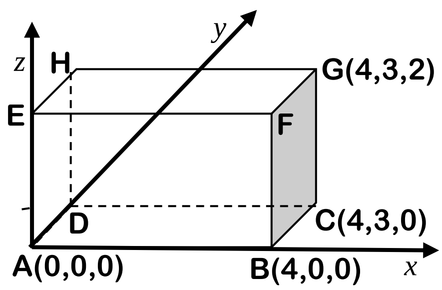
Step 13) 3D Coordinates

We have learned about (x,y) coordinates, but they only tell us about points on a flat plane or a grid in 2 dimensions. What about finding the coordinate of a point within a 3D shape or space? We can create a 3rd direction or axis at right angles to both the x-direction and y-direction, and call it the z-direction. So we can define a 3D coordinate using (x,y,z).



This is done by imagining a regular (x,y) plane flat on a piece of paper on the desk. The z-direction moves vertically up away from the desk-top's surface (or down into it for a negative z-value).

Let's look at a cuboid ABCDEFGH with A(0,0,0) at the origin & B(4,0,0) & C(4,3,0) & G(4,3,2).



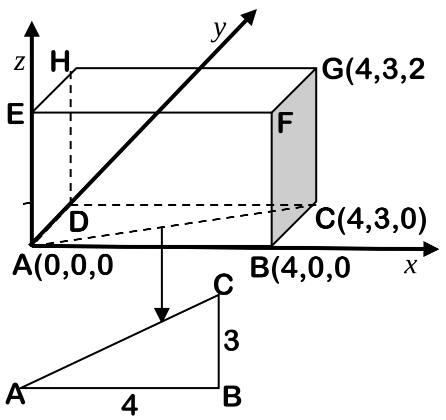
Look at the face FBCG, all the x-values are the same, 4. This is because the face is parallel to the x-axis, so any point on that face has moved the same distance from the origin in the x-direction. So F must also have x-value of 4.So F(4,?,?)

A(0,0,0) & B(4,0,0) are the same distance in the y-direction from the origin, 0. And also the same distance as F. So F(4,0,?)

In the z-direction, F is in the same plane as G(4,3,2) so F's z-coordinate is 2. So F(4,0,2)

Similarly D can be found with the same x-coordinate & z-coordinates as A(0,0,0), both 0, and the same y-coordinate as C(4,3,0), 3. So D(0,3,0).

You can find distances and midpoints in 3D, by looking at the distances between each of the x, y or z-coordinates independently (see also 3D Pythagoras and trigonometry in the angles ladder). Let's try and find the diagonal distance AC on the previous cuboid ABCDEFG.



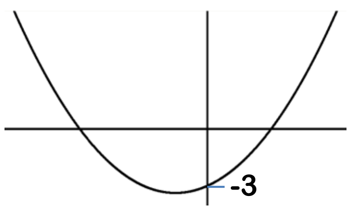
AB = dx = 4 - 0 = 4

BC = dy = 3 – 0 = 3

AC = $\sqrt{4^2 + 3^2}$

Step 14) Sketching Quadratics (a≠1)

We'll work with $y = 2x^2 + x - 3$.
The y-axis is the line $x = 0$ so we are looking for a point $(0, y)$. Let's substitute in $x = 0$ to find the y-coordinate $y = 2\text{x}0^2 + 0 - 3 = -3$, so the y-intercept is $(0, -3)$.



Now the x axis is the line $y=0$, so we can sub $y=0$ into our equation

$$0 = 2x^2 + x - 3$$

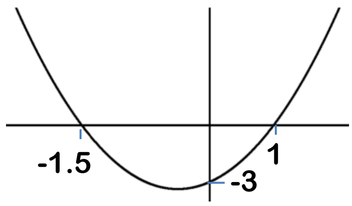
This is the only difference with step 9, this equation is harder to solve because $a \neq 1$.
Let's firstly factorise it.

$$0 = (2x + 3)(x - 1)$$

$$\textbf{so } 2x + 3 = 0 \textbf{ or } x - 1 = 0$$

$$\textbf{so } x = -1.5 \textbf{ or } x = 1$$

so the coordinates are $(-1.5, 0)$ & $(1, 0)$
And now a sketch of all these intercepts will give us a good outline of the curve.



Step 15) Sketching Quads by Completing the Square

Many quadratics can be sketched using factorisation to find the roots $(x,0)$ and substituting in $x=0$ to find the y-intercept $(0,y)$. But with completing the square we can also find the minimum point (with a positive x^2 term) or the maximum point (with a negative x^2 term).
Let's look $y = x^2 + 4x - 5$

Firstly the y-intercept is when $x = 0$
so $y = 0 + 0 - 5 = -5$
so we get $(0,-5)$ as the
then we complete the square

First guess is $(x + 2)^2 = x^2 + 4x + 4$

so $(x + 2)^2 - 9 = x^2 + 4x - 5$

now $x^2 + 4x - 5 = 0 \quad (x + 2)^2 - 9 = 0$

so $(x + 2)^2 - 9 = 0$

$(x + 2)^2 = 9$

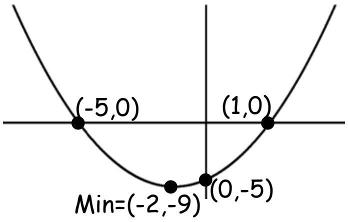
so $x + 2 = 3$ or $x + 2 = -3$

so $x = 1$or $x = -5$

Hence the roots are (1,0) & (-5,0)
And the y-intercept is (0,-5)

We know that $(x + 2)^2 - 9 = x^2 + 4x - 5$
To make this as small as possible (in other words to find it's minimum value), we need the squared bracket $(x + 2)^2$to be as small as possible. When we square it, it will become positive whether the $x + 2$ is negative or positive, so the smallest we can make $(x + 2)^2$ is 0., which is a means $x + 2 = 0$ hence x =-2.
so minimum at x = - 2
so minimum at (-2, -9)

Sketching the graph and adding these points we get.



The minimum (or maximum) is also the point where the gradient is 0, you can see that the curve is horizontal here (no step up for a step to the right). Once we have learned differentiation to find $\frac{dy}{dx}$ we can use this to find when $\frac{dy}{dx} = 0$ which must be when the graph is horizontal. This fun is all yet to come!

Step 16) Transformations of Graphs

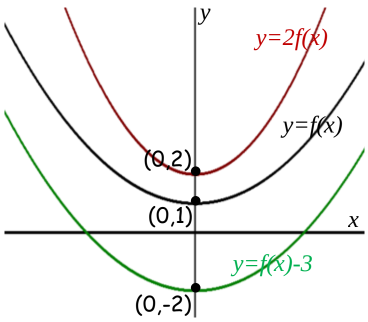
Firstly let's look at what happens when we change (or transform) graphs in the y – direction.

Let's look at a parabola called $y=f(x)$ (The black line on our sketch).

Let's sketch the line where $y=2f(x)$ in red. For the same x-value, each point has twice as big a y, so this graph will be like $f(x)$ but stretched double in the y direction,

Let's also sketch $y=f(x)-3$ in green. Every y-coordinate will be 3 less than the y-coordinate of $f(x)$, so the whole graph will simply move down 3.

$f(x)-3$ moves down 3
$2f(x)$ stretches x2 in y-direction



In general $f(x) + a$ moves $f(x)$ up a in the y direction (or down if a is negative).

And in general $bf(x)$ stretches $f(x)$ b times in the y-direction (or shrinks it if b is a fraction.

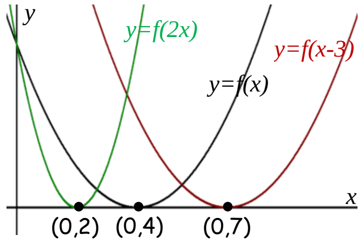
Now we will look at what happens when we make changes in the x-direction.

Again we'll start with y=f(x) sketched in black.

Firstly let's sketch $y=f(x-3)$ in red. You might assume the $x-3$ would move it left three (as negative 3 is left in the x direction). In fact we are going to put in the place of $f(x)$ the y-value from $f(x-3)$ and $x-3$ is 3 to the left of x, so we have to move it 3 right.

A similar opposite is found with multiples of x. Let's sketch $y=f(2x)$ in green. $f(2x)$ takes the y-value from $2x$, and puts it at x. In other words it halves the stretch of the curve along the x-axis. So it stretches by x½ in the x-direction.

$f(x-3)$ moves 3 right
$f(2x)$ stretches x½ in x-direction



With negative multiples of x or y
The black line is again $y=f(x)$.
$y=-f(x)$ makes every positive y-coordinate negative and vica-versa. So it flips $y=f(x)$ in the x-axis (shown in red)

Similarly , $y=f(-x)$ makes every negative x-coordinate positive and vica versa, so it flips the graph in the y-axis (shown in green)
in green

$-f(x)$ reflects in x-axis
$f(-x)$ reflects in y-axis