b) Understanding Powers

1 2 3 4 5 6 7 8

8) Surds
7) Fractional Powers
6) Negative Powers
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4) HCF & LCM
3) Prime Factorisation
2) Roots $\sqrt{}$
1) Powers x^2, x^3

b) Powers

Step 1) Powers x^2, x^3

When you repeatedly add numbers we call it multiplying.

For example 2 + 2 + 2 + 2 + 2 is called '5 multiplied by 2,' written 5 x 2. We are adding 2, 5 times, and so '5 times 2,' is a nickname for 5 multiplied by 2.

What happens if we repeatedly times a number, say 2 x 2 x 2 x 2 x 2. We call 2 multiplied 5 times '5 to the power 2,' written as 5 with a little two at the top right hand side 2^5.

Let's look at how we write down a power.

The number of times we multiply the root

3^4

The number to be repeatedly multiplied, called the root

So $3^4 = 3 \times 3 \times 3 \times 3$

1st 2nd 3rd 4th

In words 3^4 is 3 multiplied 4 times.

A number on it's own, not multiplied, can be thought of as power 1.

so $2 = 2^1$, $3 = 3^1$, $4 = 4^1$ and so on.

Power 2 and power 3 are so commonly used they have their own names.

A number to the power 2 is describes as that number squared, and a number to the power 3 is described as that number cubed.

5^2 = 5 squared = 5 x 5 = 25

7^2 = 7 squared = 7 x 7 = 49

2^3 = 2 cubed = 2 x 2 x 2 = 8

10^3 = 10 cubed = 10 x 10 x 10 = 1,000

For everything of power bigger than 3 you have to use the long phrase 'to the power of.'

3^4 = 3 to the power 4 = 3 x 3 x 3 x 3 = 81

2^7 = 2 to the power 7 = 2 x 2 x 2 x 2 x 2 x 2 x 2 = 128

Numbers that can be made by squaring a whole number are called square numbers.

Numbers that can be made by cubing a whole number are called cubic numbers.

Let's find some…

Order	Write	Calculation	Square Number
1st	1^2	1 x 1	1
2nd	2^2	2 x 2	4
3rd	3^2	3 x 3	9
4th	4^2	4 x 4	16
5th	5^2	5 x 5	25
6th	6^2	6 x 6	36
7th	7^2	7 x 7	49
8th	8^2	8 x 8	64
9th	9^2	9 x 9	81

10^{th}	10^2	10 x 10	**100**
11^{th}	11^2	11 x 11	**121**
12^{th}	12^2	12 x 12	**144**
13^{th}	13^2	13 x 13	**169**
14^{th}	14^2	14 x 14	**196**
15^{th}	15^2	15 x 15	**225**
20^{th}	20^2	20 x 20	**400**

The first 15 square numbers are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225 & the 20^{th} is 400

Now let's find a few cubic numbers too.

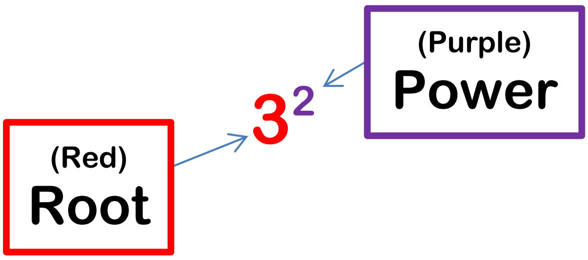
Order	Write	Calculation	Cubic Number
1^{st}	1^3	1 x 1 x 1	**1**
2^{nd}	2^3	2 x 2 x 2	**8**
3^{rd}	3^3	3 x 3 x 3	**27**
4^{th}	4^3	4 x 4 x 4	**64**
5^{th}	5^3	5 x 5 x 5	**125**
10^{th}	10^3	10 x 10 x 10	**1,000**

The first 5 cubic numbers are:

1, 8, 27, 64, 125 & the 10^{th} is 1,000

Step 2) Roots √

A root is the opposite process of a power. It asks the question, what number to I square (or cube, or power 4) to make another number.



So the answer to the question what is the root of 3^2 is 3. And the power is 2. The value of 3^2 is 3 x 3 = 9. So you could also ask the question what is the square (2^{nd}) root of 9. In other words what number do we square to make 9.

Because they are the same number written in different ways.

The square root of 9 and the square root of 3^2 must be the same thing (they are both 3). But if the question is written in the form 3^2 it is easy to see, as the 3 is squared, that the square root is 3. But if it is written in the form square root of 9, you need to work out how this looks as a root number squared.

So to find the square root of 9, we first find that 9 = 3^2, and so we know the square root is 3.

Blimey, all this writing "square root of" is making my hand ache. Mathematicians love a symbol or shortening and we have one for "square root." We put a symbol around the number (or thing) we are square rooting.

"So square root of 9" is written $\sqrt{9}$

It is like a little tick mark on the left of the number to be square rooted, with a horizontal line extended over the number to square root.

100^2 = 10,000, so $\sqrt{10{,}000} = 100$

This is read the square root of ten thousand equals 100. Not that the horizontal line at the top of the tick goes all the way over the number 10,000 just past the end of the final zero.

Let's look at one more example using the symbol for square root $\sqrt{\ }$.

Find $\sqrt{25}$. Well 5^2 = 25, so $\sqrt{25} = 5$

Or does it? Isn't $(-5)^2$ = 25 also true? So $\sqrt{25} = -5$

In fact both are correct. Because a negative times a negative makes a positive number all square roots have two answers. So $\sqrt{25} = 5 \; or -5$.

So our earlier answers were correct, but not a full answer. In fact $\sqrt{9} = 3\ or -3$, $\sqrt{10{,}000} = 100\ or -100$.

We read this "the square root of 9 is plus or mines 3." And low and behold, a shortcut has been made for writing this.

$\sqrt{9} = 3\ or -3$ can be written $\sqrt{9} = \pm 3$

$\sqrt{25} = 5\ or -5$ can be written $\sqrt{25} = \pm 5$

and $\sqrt{10{,}000} = 100\ or -100$

can be written $\sqrt{10{,}000} = \pm 100$

It is worth noting that for many numbers, the square root is not a whole number. For example $3.45^2 = 11.9025$.

So $\sqrt{11.9025} = 3.45$ This is not something to work out in your head. Luckily all most calculators have a $\sqrt{\ }$ button. There are various methods for finding decimal roots without a calculator, you can find out more about these in the decimals ladder, or in the equations step on trial and improvement.

Just as a square root asks you what the root of a square number is, a cube (3^{rd}) root asks you what the root of a cubed number is.

$2^3 = 2 \times 2 \times 2 = 8$

So the cube root of 27 must be 3. As the root, of 8, when cubing a root number, is 2. Unsurprisingly this has a symbol too. We put a little 3 inside the tick to show that it is a 3^{rd} (or cube) root. $\sqrt[3]{8} = 2$ (because $2^3 = 8$).

$\sqrt[3]{1{,}000} = 10$ (as $10^3 = 1{,}000$)

We can also have 4^{th} roots $\sqrt[4]{\ }$, which ask what root number to the power 4 makes another number. And 5^{th} roots $\sqrt[5]{\ }$ which ask which root number to the power 5 makes another number. And so on.

$\sqrt[4]{10{,}000} = 10$ (because $10{,}000^4 = 10$)

$\sqrt[5]{32} = 2$ (because $2^5 = 32$)

Just as with powers we have special names for power 2 (square) and power 3 (cube) and after that we just say power 4, power 5 and so on.

With roots, we say square root for the 2^{nd} root and cube root for the 3^{rd} root, and after that we just say 4^{th} root, 5^{th} root and so on.

Step 3) Prime Factorisation

Firstly a reminder that a factorisation is a combination of numbers that multiply to make another number.

So 2 x 3 is a factorisation of 6 (2 and 3 are both factors of 6 that form a factor pair).

A prime factorisation is one where all the factors are prime numbers.

So 4 x 5 = 20 is not a primesfactorisation of 20 because 4 is not a prime number (even though 5 is, ***all*** the factors have to be prime for it to be a prime factorisation).

But 2 x 2 x 5 = 20 is the prime factorisation of 20 because 2, 2, and 5 are all prime numbers.

Sorry to bamboozle you with language, but there is one more term to learn. In the prime factorisation 2 x 2 x 5 = 20 the prime number 2 is repeated twice, so it could be written as a power 2^2. When all repeated primes are written as powers it is called a prime power factorisation, here $2^2 \times 5 = 20$ is the prime power factorisation of 20.

Here are several factorisations (including the prime, and prime power factorisations) of 36.

Factorisation	Value	Type of Factorisation
4 x 9	36	Standard
2 x 2 x 9	36	Standard
3 x 3 x 4	36	Standard
2 x 18	36	Standard
3 x 12	36	Standard
2 x 2 x 3 x 3	36	Prime
$2^2 \times 3^2$	36	Prime Power

You can see here that there are many standard factorisations but only one prime factorisation, and one prime power factorisation.

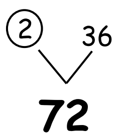
This is one of the amazing properties of prime numbers. Every number in existence can be made be multiplying a unique combination of prime numbers, and no other number can be made be multiplying that same combination of primes.

This is such an amazing, fantastic and important property of prime numbers. It is part of the reason prime numbers are so important and fascinating to mathematicians. Why a significant proportion of the history of maths study has been dedicated to prime numbers and the many amazing properties. It is well worth aligning yourself with all these historic mathematicians by discuss why the prime factorisation and prime power factorisation's of each number are unique (but you can get several standard factorisations).

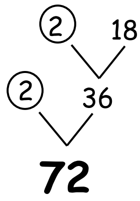
With really big numbers it can be tricky to work out the prime factors. Here is a nice method for finding them called a factorisation tree. Let's find the prime factors of 72.

You create a tree of factor pairs here, gradually working from the smallest prime number, 2, through ever larger prime numbers 3, 5, 7… seeing if they are factors. You use two branches to show where each factor pair comes from and circle any prime factors along the way.

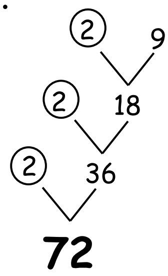
Start with the number 72, and see if the first prime number 2 goes into it. If it does draw two branches and write 2, and it's factor pair to 72 at the end of the branches. Then circle 2 as it is a prime number.



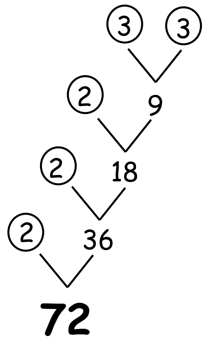
Then do the same thing for 36, see if 2 is a factor, and draw branches with its factor pair. You have to continue trying 2 until there are no more pair factors with 2, and then move onto 3 (then 5, 7…)



And try 2 again…



This time 2 is not a factor, but 3 is. It's factor pair is also 3, they are both prime so get circled.



We now have no composite numbers left to factorise. All the circled primes together have product 72!

2 x 2 x 2 x 3 x 3 = 72

And to make a prime power factorisation you just write all the 2s and 3s as powers.

2^3 x 3^2 = 72

Step 4) HCF & LCM

We learned more about factors in the last step, particularly prime factors. The Highest Common Factor (HCF) of two numbers is the highest number that is a factor of both of them. Let's find the HCF of 18 and 24.

You can use factor pairing to find all the factors of each number. Starting with searching for a factor pair for 1, then 2 and so on, writing down any factor pairs you get.

18 = 1 x 18
18 = 2 x 9

18 = 3 x 6
24 = 1 x 24
24 = 2 x 12
24 = 3 x 8
24 = 4 x 6

Factors of 18 are 1, 2, 3, 6, 9, & 18

Factors of 24 1, 2, 3, 4, 6, 8, 12 & 24

Common means the same, as in the two lists of factors have that number in common. 1, 2, 3 and 6 are all common factors of 18 and 24.

But the largest number that is in both lists, that is a factor of both 18 and 24 is 6, so the HCF of 18 & 24 = 6

Another way to find the HCF is using the prime power factorisation of each number. Let's write down the prime power factorisation of 18 and 24.

18 = 2 x 3 x 3

24 = 2 x 2 x 2 x 3

The prime factors common to both numbers, will give us the prime factors of the HCF.

We couldn't expect another 2, for example to be in the HCF's prime factorisation, because only one 2 is in the prime factorisation of 18. If the HCF had two 2s in its prime factorisation we'd need two 2s in both the numbers themselves.

Similarly, although there are two 3s in the prime factorisation of 18, there is only 1 in the prime factorisation of 24, so there can only be one in a number that is a factor of both.

HCF = 2 x 3 = 6

The Lowest Common Multiple (LCM) of two numbers is the smallest number that is a multiple of both. Let's list multiples of both and see which is the lowest common multiple.

Multiples of 18: 18,36,54, 72, 90,108,126, 144

Multiples of 24: 24, 48, 72, 96, 120, 144, 168

72 and 144 are both common multiples of 18 and 24, but 72 is the lowest of all their common multiples.

LCM of 18 & 24 = 72

It is interesting to note that there will be infinitely many more common multiples of both numbers, as any multiple of a common multiple will also be a common multiple.

LCM can also be found using the prime power factorisation of each number.

18 = 2 x 3 x 3

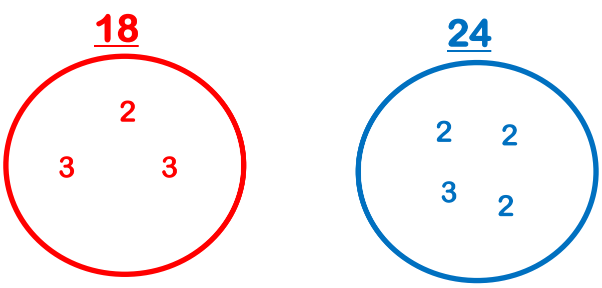
24 = 2 x 2 x 2 x 3

If a number is a multiple of 18, it must have at least one 2 in its list of prime factors. If it is a multiple of 24 then it must have at least three 2s in its list of prime factors. So overall it only needs three 2s to be a multiple of both (the 3 twos in 24 provide also at least the 1 needed for 18)

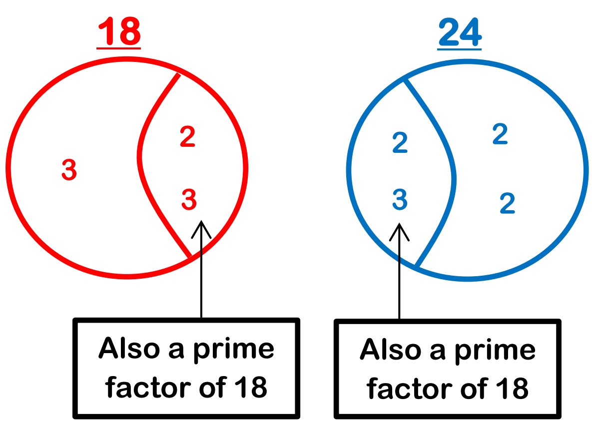
Similarly we'll need only two 3s, as this will also cover the single 3 needed for it to be a multiple of 24.

LCM = 2 x 2 x 2 x 3 x 3 = 72

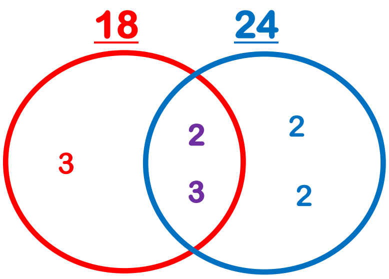
A nice way to bring all this together is to use the inner circle parts of a Venn diagram. I'll put the prime factors of 18 in a red circle and the prime factors in a blue circle.



Next we can separate out the prime factors in each circle to group those that are also prime factors of the other number.



Now if we slide the two together we get a middle section with the common prime factors that you don't need to draw both times. We can label these common prime factors in purple (which is a mix of blue and red).



Now the common prime factors in the middle purple bit can be multiplied to give the highest common factor. All the numbers here can be multiplied to give the lowest common multiple (any repeated common prime factors have been removed in the purple section, so we don't get the product, but a lower common multiple if there is one).

So the HCF = 2 x 3 = 6

And the LCM = 2 x 2 x 3 x 2 x 2 = 72

from 18   from both   from 18

Or put another way.

18

2 x 2 x 3 x 2 x 2 = LCM = 72

24

It is worth noting from this, that if you multiply the LCM by HCF, you get the multiple of both the numbers.

$$\overbrace{2 \times 2 \times 3}^{18} \times \underbrace{2 \times 3 \times 2 \times 2}_{24} = \text{product}$$

This can be summerised for any two numbers as:

Product = HCF x LCM

Step 5) Power Rules

Let's look at what happens when you multiply two numbers that are a power of the same root.

$3^5 \times 3^2$

We can expand this as follows

multiplied 5 times   2 times   total times

1 2 3 4 5   1 2   = 5 + 2 = 7

$$\underbrace{3 \times 3 \times 3 \times 3 \times 3}_{\text{expansion of } 3^5} \times \underbrace{3 \times 3}_{\text{expansion of } 3^2} = 3^{(5+2)} = 3^7$$

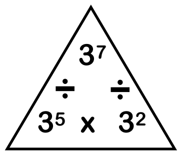
In general however many times you multiply in the first and second powers with same root, you will add together the number of mutiplyings when putting them together. Some people like to write this using algebra, don't worry if y ou haven't studied algebra yet, take a read and if it doesn't make sense remember the process above and move on to dividing power of same root.

$$x^a \times x^b = x^{a+b}$$

Now let's look at dividing powers of the same base.

Putting the above $3^5 \times 3^2 = 3^7$ in a multiplication triangle we can see also that

$$\frac{3^7}{3^5} = 3^2$$



So we can see why in terms of x and ÷being opposites, let's look at it directly by expanding the powers as we did for multiplying powers of the same root.

$$\frac{3^7}{3^5} = \frac{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3 \times 3}$$

We can use the properties of multiplying and dividing fractions to split this into several multiplied fractions

$$3 \times 3 \times \frac{3}{3} \times \frac{3}{3} \times \frac{3}{3} \times \frac{3}{3} \times \frac{3}{3}$$

But each $\frac{3}{3}$ has value 1, so

$$3 \times 3 \times \underbrace{\frac{3}{3} \times \frac{3}{3} \times \frac{3}{3} \times \frac{3}{3} \times \frac{3}{3}}_{\text{removing five 3s from the top and bottom}}$$

$= 3 \times 3$

$= 9$

Let's put it all together

$$\frac{3^7}{3^5}$$

$$= \frac{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3 \times 3}$$

$$= 3 \times 3 \times \frac{3}{3} \times \frac{3}{3} \times \frac{3}{3} \times \frac{3}{3} \times \frac{3}{3}$$

$= 3 \times 3$

$= 3^2$

$= 9$

So $\frac{3^7}{3^5} = 3^2 = 9$

In general, however many 3s are on the bottom, we create that many $\frac{3}{3}$s, removing the same number of 3s from the top (as each $\frac{3}{3} = 1$). Hence we are subtracting the power of the denominator from the power of the numerator.

So $\frac{3^7}{3^5} = 3^{7-5} = 3^2 = 9$

Finally let's look at a power of a power.

$$(5^3)^2 = 5^3 \times 5^3 = 5^{3+3} = 5^{3\times2} = 5^6$$

$$(5^3)^3 = 5^3 \times 5^3 \times 5^3 = 5^{3+3+3} = 5^{3\times3} = 5^9$$

$$(5^3)^4 = 5^3 \times 5^3 \times 5^3 \times 5^3 = 5^{3+3+3+3} = 5^{3\times4} = 5^{12}$$

Because powers are repeated multiplying, this is like the mutlipling powers rule, but we keep adding, here, 3s to the power, for each increasing power of 5^3. SO in general

$$(5^3)^b = 5^{3b}$$

And this will work with any base or root, so

$$(x^a)^b = x^{ab}$$

In other words with a power of a power, we multiply the powers.

To summarise:

To times powers we add them	$x^a \times x^b = x^{a+b}$
To divide powers we subtract them	$\frac{x^a}{x^b} = x^{a-b}$
For a power of a power we times them	$(x^a)^b = x^{ab}$

Step 6) Negative Powers

Let us first look at what happens when we add 1 to a power of 2.

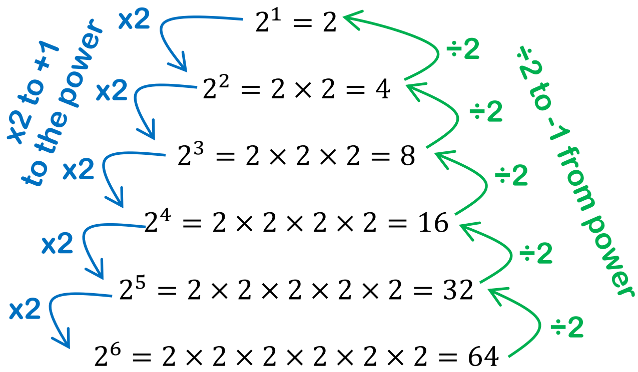
$2^{3+1} = 2^3 \times 2^1 = 2^3 \times 2 = (2 \times 2 \times 2) \times 2$

So when we add one to a power of 2, we are multiplying by two. And the opposite must also be true. When we subtract one from a power of 2, we are dividing by 2.

$2^{4-1} = \frac{2 \times 2 \times 2 \times 2}{2} = 2 \times 2 \times 2 \times \frac{2}{2} = 2 \times 2 \times 2 = 2^3$

If we understand subtracting from the power as dividing because it is the opposite of adding to the power, which is multiplying, then some huge results follow.

Let's look at this in another way, using repeated increasing of the power of 2.



This same explanation can be used for powers of any number. Let's try this all again with powers of 10

x10 to +1 to the power

÷10 to -1 from power

$$10^1 = 10$$

x10 ÷10

$$10^2 = 10 \times 10 = 100$$

x10 ÷10

$$10^3 = 10 \times 10 \times 10 = 1{,}000$$

x10 ÷10

$$10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$$

Having looked in detail at why subtracting 1 from a power is like dividing by 10, we can now look at what happens if you keep subtracting 1 from positive powers, past 0, into negative powers.

x10 to +1 to the power

÷10 to -1 from power

$$10^{-4} = \frac{1}{10^4} = \frac{1}{10{,}000}$$

÷10

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000}$$

÷10

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$$

÷10

$$10^{-1} = \frac{1}{10}$$

÷10

$$10^0 = \frac{10}{10} = 1$$

÷10

$$10^1 = 10$$

÷10

$$10^2 = 10 \times 10 = 100$$

x10 ÷10

$$10^3 = 10 \times 10 \times 10 = 1{,}000$$

x10 ÷10

$$10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$$

So we can see that just as

$10^x = 10 \times\times 10 \times 10 \ldots x$ times

$10^{-x} = 1 \div 10 \div 10 \div 10 \ldots x$ times $= \frac{1}{10^x}$

Let's check this with negative powers of 2, but without all the arrows, siumply by dividing by 2 each time (we'll start at 2^3).

$$2^3 = 2 \times 2 \times 2 = 8$$

$$2^2 = 2 \times 2 = 4$$

$$2^1 = 2$$

$$2^0 = \frac{2}{2} = 1$$

$$2^{-1} = 1 \div 2 = \frac{1}{2}$$

$$2^{-2} = 1 \div 2 \div 2 = \frac{1}{2^2} = \frac{1}{4}$$

$$2^{-3} = 1 \div 2 =\div 2 \div 2 \frac{1}{2^3} = \frac{1}{8}$$

$$2^{-4} = 1 \div 2 \div 2 \div 2 \div 2 = \frac{1}{2^4} = \frac{1}{16}$$

$$2^{-5} = 1 \div 2 \div 2 \div 2 \div 2 \div 2 = \frac{1}{2^5} = \frac{1}{32}$$

$$2^{-6} = 1 \div 2 \div 2 \div 2 \div 2 \div 2 \div 2 = \frac{1}{2^6} = \frac{1}{64}$$

So in general a negative power is dividing by the base a number of times, so we end up with 1 over that positive power.

Step 7) Fractional Powers

Let's look at $100^{\frac{1}{2}}$.

We'll need the power rule for multiplying for this step

$$x^a \times x^b = x^{a+b}$$

We'll try squaring $100^{\frac{1}{2}}$

$$100^{\frac{1}{2}} \times 100^{\frac{1}{2}} = 100^{\frac{1}{2}+\frac{1}{2}} = 100^1 = 100$$

And as

$$100^{\frac{1}{2}} \times 100^{\frac{1}{2}} = 100$$

Then $100^{\frac{1}{2}} = 10$ because 10 x 10 = 100

With a power ½ you are looking for a number which multiplies by itself to find the root. In other words it is the square root.

$$100^{\frac{1}{2}} = \sqrt{100} = 10$$

In general, in terms of x

$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$$

or in short

$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x$$

So $x^{\frac{1}{2}}$ is the thing that when we multiply it by itself makes x, this is the very definition of square root. So $x^{\frac{1}{2}} = \sqrt{x}$

We can work with this similarly with $8^{\frac{1}{3}}$

$$8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 8^1 = 8$$

And as 2 x 2 x 2 = 8

$8^{1/3} = 2$

In other words $8^{\frac{1}{3}}$ is the thing you cube to make 8, that is the cube root of 8.

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

In general $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x$

so $x^{\frac{1}{3}}$ is the thing that when you cube it, makes x, in other words the cube root of x.

$$x^{\frac{1}{3}} = \sqrt[3]{x}$$

In the most general terms

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

Because when you times $x^{\frac{1}{n}}$ by itself n times, you always get x^1which is just x.

So now we have $x^{\frac{1}{2}} = \sqrt{x}$

And also $x^{\frac{1}{3}} = \sqrt[3]{x}$

And in general $x^{\frac{1}{n}} = \sqrt[n]{x}$

Let's look at this with some examples. We will build up the complexity, and also for each one, see what happens if we take the same fractional power, but as a negative fraction, to mix this all in with the previous step.

$$64^{\frac{1}{2}} = \sqrt{64} = 8$$

$$64^{-\frac{1}{2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}$$

$$1{,}000^{\frac{1}{3}} = \sqrt[3]{1{,}000} = 10$$

$$1{,}000^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{1{,}000}} = \frac{1}{10}$$

$$27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$$

This reverses the power of a power rule…

$x^{\frac{a}{b}} = \left(x^{\frac{1}{b}}\right)^a$ and also $x^{\frac{a}{b}} = (x^a)^{\frac{1}{b}}$

So we could have sued

$27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2$ or $27^{\frac{2}{3}} = (27^2)^{\frac{1}{3}}$

The reason we used the fractional power first is it created a smaller number to then put to a whole number power. We could square 27 first, but we then have a very large number to find the cube root of. It is possible, but harder.

And yes, we need to see how it looks with power $-\frac{2}{3}$ too!

$$27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{\left(27^{\frac{1}{3}}\right)^2} = \frac{1}{\left(\sqrt[3]{27}\right)^2} = \frac{1}{3^2} = \frac{1}{9}$$

We'll take one more example of a complex fraction (top and bottom not 1) as they are seriously hard to get your head around (keep trying them).

$$4^{\frac{5}{2}} = \left(4^{\frac{1}{2}}\right)^5 = \left(\sqrt{4}\right)^5 = 2^5 = 32$$

And finally with power $-\frac{5}{2}$

$$4^{-\frac{5}{2}} = \frac{1}{\left(4^{\frac{1}{2}}\right)^5} = \frac{1}{\left(\sqrt{4}\right)^5} = \frac{1}{2^5} = \frac{1}{32}$$

Step 8) Surds

A surd is a number which is a square root, written in root form $\sqrt{of\ a\ number}$

We will start with a reminder of the first two power rules…

To times powers we add them	$x^a \times x^b = x^{a+b}$
To divide powers we subtract them	$\dfrac{x^a}{x^b} = x^{a-b}$

$$x^a \times x^b = x^{a+b}$$

We could experiment also with what happens if we multiply numbers of the same power but with different roots or bases. For example

$$3^2 \times 7^2 = 3 \times 3 \times 7 \times 7 = 3 \times 7 \times 3 \times 7 = (3 \times 7)^2$$

So we can extend this rule to make one that says when we multiply the same power of different bases, we multiply the bases.

$$a^x \times b^x = (ab)^x$$

The reason we are doing all this, is because that power might be a fractional one like ½ (square root) or ⅓ (cube root).

Let's look at this rule when x = ½

$$a^{\frac{1}{2}} \times b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}$$

And if we put that in square root notation

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

So for example $\sqrt{75}$ could be written as

$$\sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$$

5 lots of $\sqrt{3}$ feels simpler than $\sqrt{75}$

It is important not to confuse

$5\sqrt{3}$ **which is** $5 \times \sqrt{3}$**, 5 lots of** $\sqrt{3}$

and $\sqrt[5]{3}$ which is the 5th root of 3 (the number I multiply 5 times to make 3)

Here is another example with some tips on how we spot the way to simplify a surd.

Let's try and simplify the surd $\sqrt{60}$

Using $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

We can split $\sqrt{60}$ up in several different ways

$$\sqrt{60} = \sqrt{3 \times 20} = \sqrt{3} \times \sqrt{20}$$

$$\sqrt{60} = \sqrt{5 \times 12} = \sqrt{5} \times \sqrt{12}$$

$$\sqrt{60} = \sqrt{6 \times 10} = \sqrt{6} \times \sqrt{10}$$

$$\sqrt{60} = \sqrt{4 \times 15} = \sqrt{4} \times \sqrt{15}$$

But only a square root of a square number can be simplified out of a surd. So the only useful rearrangement of $\sqrt{60}$ is $\sqrt{4} \times \sqrt{15}$

as $\sqrt{4} = 2$

so $\sqrt{60} = \sqrt{4} \times \sqrt{15} = 2\sqrt{15}$

But how do we know which type of factorisation to use. The trick is always to look for square factors, as they will then simplify out of a surd.

It is worth noting that we could further simplify the $\sqrt{20}$ from

$$\sqrt{60} = \sqrt{3 \times 20} = \sqrt{3} \times \sqrt{20}$$

Hence

$$\sqrt{60}$$

$$= \sqrt{3 \times 20}$$

$$= \sqrt{3} \times \sqrt{20}$$

$$= \sqrt{3} \times \sqrt{5 \times 4}$$

$$= \sqrt{3} \times \sqrt{5} \times \sqrt{4}$$

$$= \sqrt{3 \times 5} \times 2$$

$$= 2\sqrt{15}$$

but this is soooo long winded. It is much easier just to spot that 4 is a square factor of 60 in the first place. It is similar to simplifying a fraction or a ratio, we can do it in lots of little steps, but easiest to find the largest factor of the numerator and the denominator (or all parts of the ratio) and divide by that first. So here, rather than finding a factor of 60, which can then be factorised into a

square factor, try and find the square factor(s) of 60 from the start.

One more example…

$$\sqrt{200}$$

$$= \sqrt{100 \times 2}$$

$$= \sqrt{100} \times \sqrt{2}$$

$$= 10\sqrt{2}$$

Though there are lots of factors of 200, and even more than one square factor, 4, 25 and 100, we went straight for the largest square factor, 100.

This can help us in simplifying much more complex things…

$$(2 + \sqrt{18})(5 + 2\sqrt{18})$$

firstly we'll multiply out the brackets

	2	$\sqrt{18}$
5	10	$5\sqrt{18}$
$2\sqrt{18}$	$10\sqrt{18}$	36

$$= 10 + 36 + 5\sqrt{18} + 10\sqrt{18}$$

$$= 46 + 15\sqrt{18}$$

$$= 46 + 15\sqrt{9 \times 2}$$

$$= 46 + 15 \times \sqrt{9} \times \sqrt{2}$$

$$= 46 + 45\sqrt{2}$$

Let's look at different roots or bases of the same power with division, and see if we can find a general rule

We'll look at $\frac{5^3}{4^3} = \frac{5\times5\times5}{3\times3\times3} = \frac{5}{3} \times \frac{5}{3} \times \frac{5}{3} = \left(\frac{5}{3}\right)^3$

So

$$\frac{5^3}{4^3} = \left(\frac{5}{3}\right)^3$$

In general

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

So $\frac{\sqrt{45}}{\sqrt{3}} = \sqrt{\frac{45}{3}} = \sqrt{15}$

And $\frac{7\sqrt{75}}{\sqrt{3}} = 7\sqrt{\frac{75}{3}} = 7\sqrt{25} = 35$

Very useful to know that something so complex as $\frac{7\sqrt{75}}{\sqrt{3}}$ can be simplified to 35!

One final note on this, and that is that mathematicians particularly like to simplify the bottom part of fractions. This means have them rational (a fraction or whole number, rather than irrational, ie a surd).

If we have a surd in our denominator we can rationalise it simply by timesing by a fraction of value 1, with this surd as the top and the bottom, let's look at $\frac{7}{\sqrt{3}}$

$$\frac{7}{\sqrt{3}} = \frac{7}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

We call this rationalising the denominator. Here's another one

$$\frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

One final note is that even with complex roots, that include both a surd and an integer there is a clever trick for rationalising the denominator. We use the difference of squares, and the fact that a square root, squared, is no longer a surd. We take fraction of value one to rationalise an integer ***plus*** a surd, to be the integer ***minus*** the surd over itself.

$$\frac{2}{1 + \sqrt{3}} = \frac{2}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$

Tops first

	1	$-\sqrt{3}$
2	2	$-2\sqrt{3}$

$$2(1 - \sqrt{3}) = 2 - 2\sqrt{3}$$

Then denominators

	1	$\sqrt{3}$
1	1	$\sqrt{3}$
$-\sqrt{3}$	$-\sqrt{3}$	-3

$$(1 + \sqrt{3})(1 - \sqrt{3}) = 1 - 3 + \sqrt{3} - \sqrt{3} = -2$$

ta da, the $+\sqrt{3}$ & $-\sqrt{3}$ and we have no surds on the bottom (it is rationalised)

$$\frac{2}{1+\sqrt{3}}$$

$$=\frac{2}{1+\sqrt{3}}\times\frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$=\frac{2-2\sqrt{3}}{-2}$$

$$=\frac{2}{-2}-\frac{2\sqrt{3}}{-2}$$

$$=-1+\sqrt{3}$$

So

$$\frac{2}{1+\sqrt{3}}=-1+\sqrt{3}$$

A lot simpler, and with no irrational denominator as required.