p) Understanding TRRE

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p) TRRE

Step 1) Left and Right

This ladder is all about changing shapes. The fancy name for this is to transform shapes, or transformations. Here we describe it using the four main transformations which begin with the letters T, R, R and E (see step 3!

The very first thing we need to understand to change things is about direction, and the first directions we learn come from splitting ourselves and our world into two equal halves down the middle. We call the two sides of the world and ourselves the left and the right.

There are two common ways to remember which direction we call left (⟵) and which direction we call right (⟶).

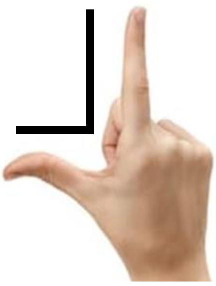
Left Hand

Right Hand

First try to make an L shape with your thumb and forefinger (with the back of the hand facing you) – you can only do it with the left hand!

Left Hand

Right Hand



2nd (and this only works if you are one of the 90% of people who write with their right hand), see which hand you write with.

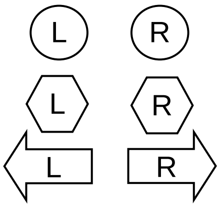
Left Writing Hand

10% of people

Right Writing Hand

90% of people

TO finish off with the idea of left and right, let's label some pairs of shapes with L for the shape on the left hand side and R for the shape on the right hand side.



Step 2) Home and Street Directions

The first type of change we need to understand is how to change your own position in your world. In other words how to move from where you are (say in your maths classroom) to where you want to be (say the toilet). We call the instructions, or steps you follow to get where you want to go directions. Whether around a building, or around the streets, directions usually are made up of turning right (towards your right hand) left (towards your left hand) or walking straight ahead. They might also include crossing roads, and going up and down stairs or in lifts.

Here are the directions from my classroom to the toilet.

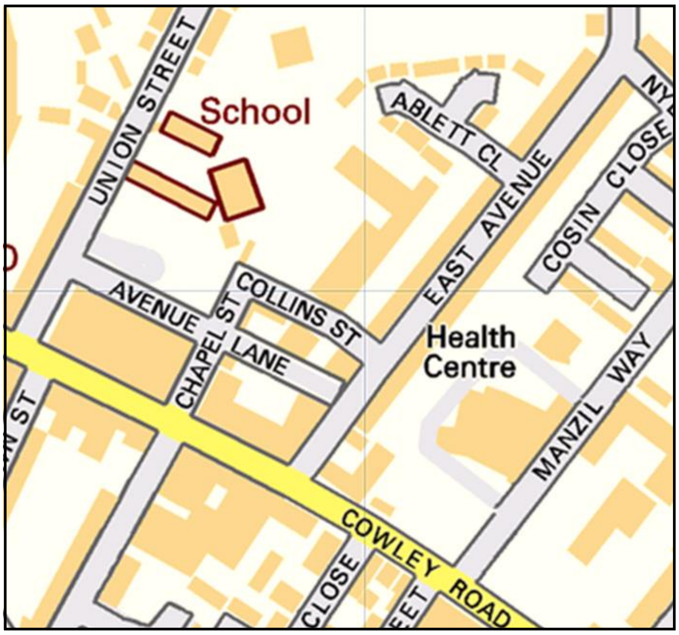
Instruction	Meaning
1) Go out of the door & turn right	Walk to the classroom door and walk through the door. Look at your hands and decide which your right hand is, turn in that direction.
2) Go to the end of the corridor	Walk along the corridor until you can't walk any further.
3) Turn Left	Look at your hands and decide which your left hand is, turn in the direction.
4) It's the second door on the right	Look along the corridor, using your left and right hand directions, decide which is the left and the right walk. Walk along the corridor counting the doors on the right hand wall. Go past the first door, the second door is the toilet door.

Directions around streets in a town work in a similar way as in buildings. You need to know your left and right, but now instead of walking along corridors and up and down stairs, you are walking along roads or streets and crossing at traffic lights and so on.

Here are instructions to get from East Oxford Primary school to the local health centre.

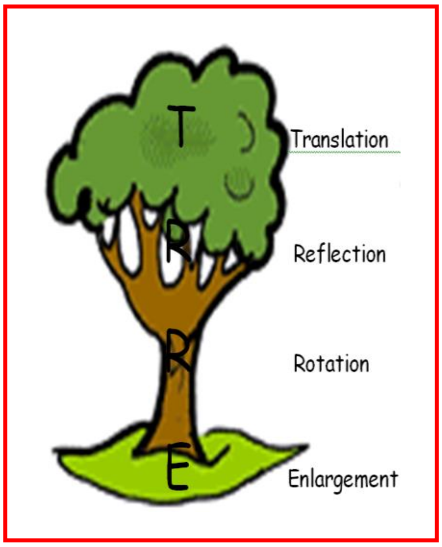
Instruction	Meaning
1) Turn left out of the school and walk to the end of Union Street.	Looking out from the school entrance, turn towards your left hand and walk down the road (called Union Stree) until you reach the end (in this case the end of the road is another larger going across the end of Union Street).
2) Turn left onto Cowley Road.	Turn towards your left had so you are facing along the Cowley Road.
3) Take the 3rd left, into Manzil Way.	Walk along the road and count the turnings on the left. The 3rd one is where you turn left onto Manzil Way.
4) The Health Centre is on the left.	Walk along Manzil Way looking for the health centre on the side of your left hand.

You can try and follow the map below, but it is easier to learn directions in an actual building or town, where you can move about and face in different directions, turning left or right, On the map, you have to imagine you that you are standing at a particular place, and imagine which way you would have to turn. More on this can be found in step 6 of the constructions ladder (m).



Step 3) The Transformation TREE

What we call TRRE on the maths ladder map, is normally called transformations in maths classrooms. To transform means to change, just like the well-loved comic and cartoon characters, the transformers. They changed (or transformed) between being robots, and being cars, bikes and boats. In maths we sometimes want to change a shape or point, perhaps moving it, turning it, making it larger or smaller, or even flipping the left and rights sides of it. TRRE is a way of remembering the four main types of transformation; translation, reflection, rotation and enlargement.

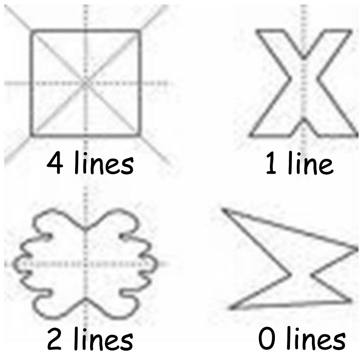


Here is what each of them mean.

Transformation	What it Means	Key Words
Translation	To Move	Up & Down Left & Right Vector
Reflection	To Flip or Mirror	Line of Reflectional Symmetry, Object & Image, Mirror Line
Rotation	To Turn	Order of Symmetry, Angle of Turn, Direction of Turn, Centre of Turn
Enlargement	Make Bigger or Smaller	Scale Factor, Centre of Enlargement

4) Reflectional Symmetry

Lines of symmetry, are lines where the images on either side of the line are mirror images, the same image but flipped left for right. Some shapes have no lines where of symmetry and others have several. A circle has infinitely many lines of symmetry, wherever you draw a mirror line through its centre, you have a line of symmetry.

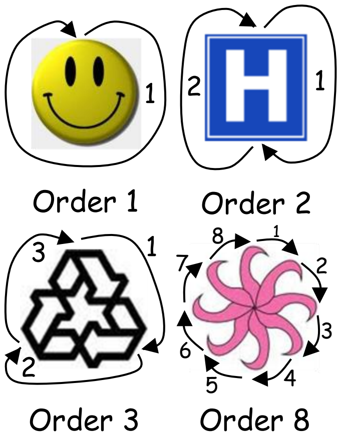


Looking at the shape with 1 line of symmetry, we might think it has two, as it is like an x-shape. But the "legs" of the shape area wider than the "arms" of the shape, so it can't be reflected top to bottom, only left to right.

Try putting a mirror against each line and seeing if the shape looks exactly the same as without the mirror.

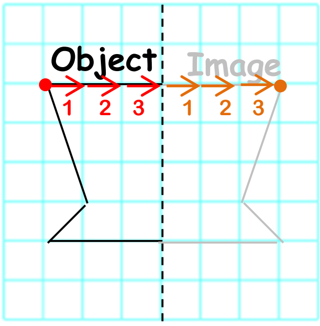
5) Rotational Symmetry

Rotational symmetry is when a shape is the same after having been rotated. Every shape is the same after you turn it a full turn (360°). Some shapes look the same at part ways through a full turn. The number of times the shape looks the same once it is back "upright" after a full turn is called the order of rotational symmetry.

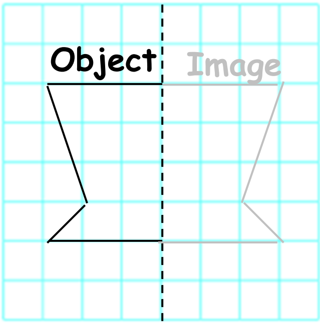


Step 6) Reflections on Squared Paper

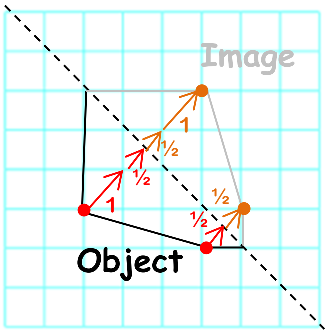
We call the shape we are going to reflect the object. The shape we draw after reflection is called the image. This is usually done on squared paper. We trace each vertex of the shape along the grid lines to the mirror line, and then continue the same distance in the opposite side of the mirror line.



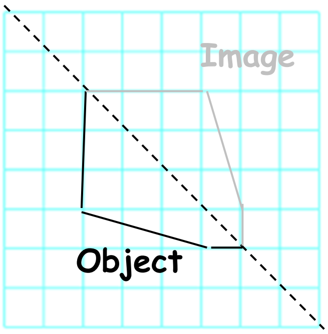
Doing this for all the vertices, we can join them up to make the image.



The process is the same if the mirror line is diagonal, we just need to ensure we go direct (at right angles) to the mirror line, and the same distance the other side. Note that some of the measures only cross a half square diagonal, this is common with 45^0 mirror lines.

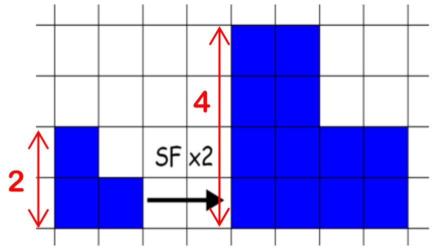


We don't draw the construction arrows, they are just for your understanding of how we find the image. It will actually look like this.

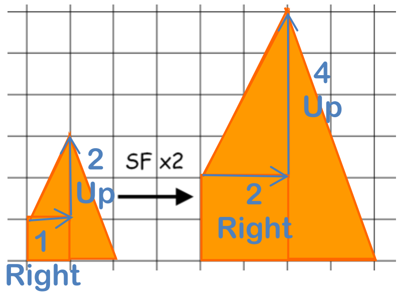


Step 7) Enlargement

An enlargement is where you make a shape bigger (or smaller – see step 11). The number of times longer you make the sides of the shapes is called the scale factor or SF for short. To enlarge a shape on square paper where all the sides are horizontal or vertical (in other words in line with the grid lines) you simply count each side, and multiply the length of the side by the scale factor for the corresponding side on your enlarged shape.



Here the left side was 2 squares high, so as the scale factor is2, we double this, and the left side in the enlarged shap becomes 4 high. Similarly the right hand side goes from being 1 high, to being 2 high, and so on, When some sides are diagonal it can be harder. In the example below, the left is 1, so it becomes 2 after transformation. The base is 2 so it becomes 4 after transformation. The diagonals can be dealt with by measuring how far horizontally then how far vertically, and then doubling this.



So 1 right and 2 up, becomes 2 right and 4 up (both distances have been doubled).

The concept of doing two different movements to represent a diagonal movement – that is some left and right movement, and then some up or down movement, instead of one diagonal movement – is defined more simply in the use of vectors in the next step.

Step 8) Translations as Vectors

A translation is changing the position of a point or shape. It is a hard word to remember for most people, and can also be confused with transformation. Transform is general word for change that includes translation, (movement), reflection (in a mirror line), rotation (turning) and enlargement (changing the size). But translation is changing the position of a shape. Some people find it helpful to remember that to translate is to move between languages, and one might move country to get the best experience of speaking or learning that language.

We can simply move points or shapes left, or right, or up or down. These simple horizontal and vertical movements are the building blocks of translation. But most translations are diagonal, and need a combination of a horizontal (left or right) movement and a vertical (up or down) movement.

We always do these movements horizontal first, and vertical second, just like with coordinates. y is a vertical movement and is done second. In fact a vector even looks like a coordinate, just one that is written vertically.

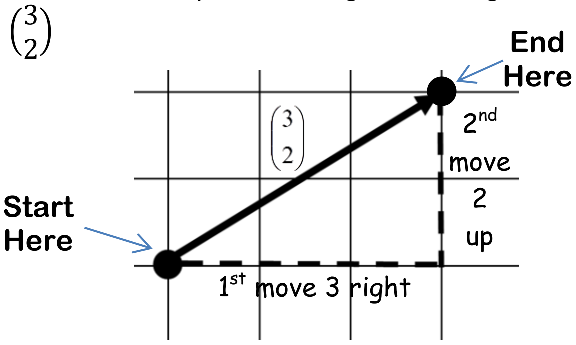
Coordinate (x, y) and Vector $\binom{x}{y}$

In both the coordinate and the vector the x is a horizontal movement and is done first.

And the y is the vertical movement (up and down) and is done after the horizontal movement.

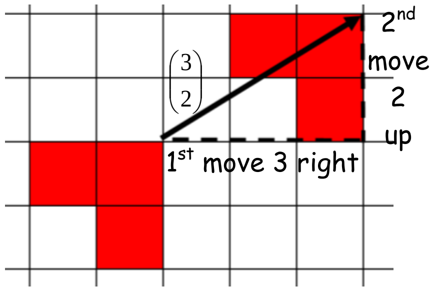
However there is one massive difference between a coordinate and a vector (apart from the fact that the brackets are written differently). A coordinate represents a position relative to the origin, the centre of the grid (0,0). The horizontal and vertical "movements," are from the origin and are done only to find its position relative to the origin. But a vector can represent a movement FROM ANYWHERE. It has both a size (the length of the line) and a direction (the angle of the line) and both of these are represented by the two numbers representing the horizontal and the vertical movement.

Let's move a point on a grid through vector $\binom{3}{2}$



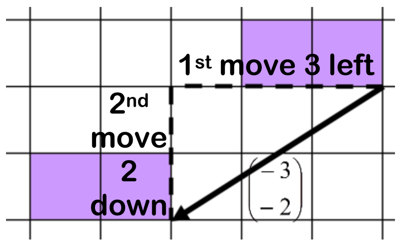
Now let's move a whole shape through the same vector $\binom{3}{2}$.

This shape has 6 vertices. We could just translate each vertex through the vector $\binom{3}{2}$ and then join up the edge lines between the vertices in their new position. However, because moving a shape doesn't change anything other than its position, its size and rotation stays the same. So we can just move one vertex (here the top right) and fill in the shape relative to the new position of this one vertex.



Negative values in the vector brackets just mean movements left (for x) or down (for y).

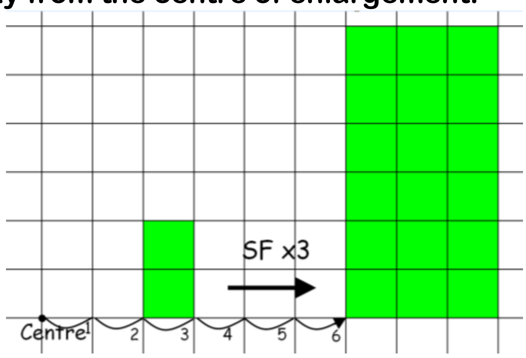
Let's move this shape through vector $\binom{-3}{-2}$



Step 9) Enlargement from a Centre

Let's enlarge this 1 wide, 2 high green rectangle by SF 3 from the centre shown. We know we should end up with a 3 wide, 6 high rectangle, as those are 3 times the original dimensions and it is SF 3.

The bottom left vertex is 2 unit horizontally to the right, from the centre of enlargement. So the new bottom left vertex must be 6 (SF3 x 2) away from the centre of enlargement.



We then simply fill in the enlarged shape.

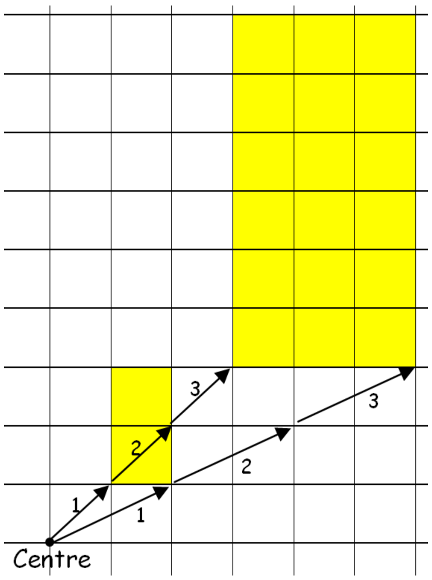
Now let's try one where the vertices are all diagonal to the centre, let's enlarge this yellow rectangle, that is 1 unit wide and 2 units high, still with SF 3, but with the centre of enlargement in a different place.

Here the bottom left hand vertex is $\binom{1}{1}$ from the centre of enlargement.

The bottom right hand vector is $\binom{2}{1}$ from the centre of enlargement.

We have to triple both of these (as it is SF 3) so the new bottom left hand vertex must be $\binom{1}{1}+\binom{1}{1}+\binom{1}{1}=3\times\binom{1}{1}=3\binom{1}{1}=\binom{3}{3}$ from the centre. Let's label each vector as 1, 2, 3, so you can see that we have done the vector 3 times.

The new bottom right hand vertex needs to be $\binom{2}{1}+\binom{2}{1}+\binom{2}{1}=3\times\binom{2}{1}=3\binom{2}{1}=\binom{6}{3}$ from the centre of enlargement.



We then draw in the rest of the shape.

We don't need to reposition all the vertices, just one will do, then enlarging the shape from that vertex. We did 2 here just to give two examples of how we can use vectors to do this.

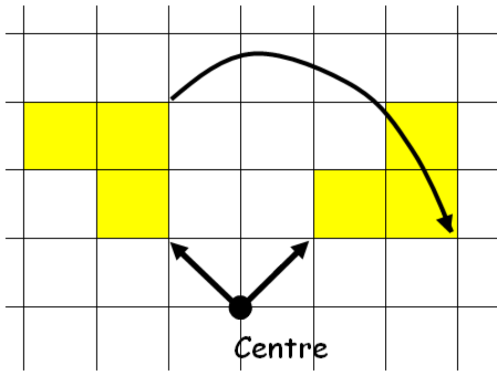
Step 10) Rotation from a Centre

Just as we can enlarge a shape so that each vertex has multiplied its distance from the centre of enlargement SF times, we can rotate a shape so that each vertex rotates a given angle about a centre of rotation, whilst still being the same distance away.

Let's rotate the yellow shape on the left 90^0 clockwise about the centre of rotation.

The bottom right vertex is $\binom{-1}{1}$ and after rotation it is $\binom{1}{1}$.

The top right vertex of the original shape is $\binom{-3}{1}$, and rotating it is $\binom{3}{1}$.



You can use tracing paper to do this, putting a sharp pencil or the point of a pait of compasses at the centre and seeing where it ends up when you turn it. Or you can look vertex by vertex above.

Or you can do maths A-level and learn about rotations using matrices (The vector brackets are a very small matrix!)

it is worth noting that rotating a shape 180^0 clockwise or anticlockwise has the same effect. They are equivalent rotations.

Any opposite direction rotations that add up to 360^0 will be equivalent.

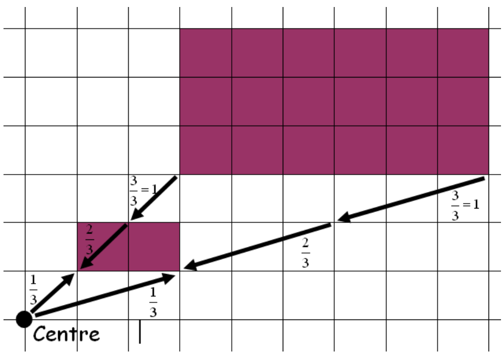
For example 90^0 clockwise is equivalent to 270^0 anti-clockwise.

Step 11) Enlargement with Negative and Fractional Scale Factors

Let's look at what happens if the scale factor is a fraction less than 1.

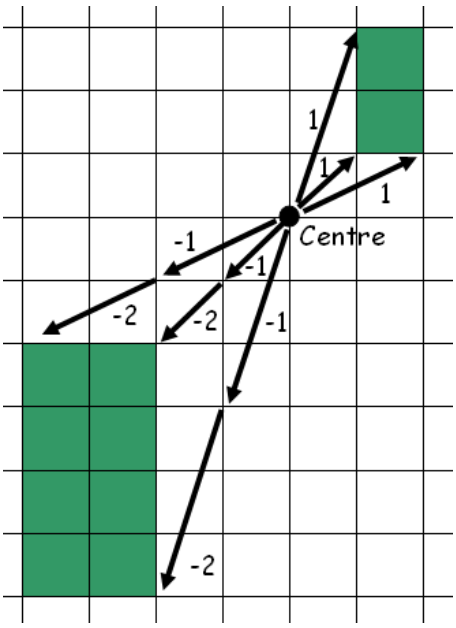
If a SF 2 makes the shape twice as big, then a SF ½ will make the shape ½ as big.

Enlarge this 6 by 2 rectangle by scale factor ⅓.



So the bottom right corner was $\begin{pmatrix}9\\3\end{pmatrix}$ from the centre, so the new bottom right corner needs to be 1/3 of that, which is $\begin{pmatrix}3\\1\end{pmatrix}$. Similarly the new bottom left corner is one third of $\begin{pmatrix}3\\3\end{pmatrix}$ from the centre, which is $\begin{pmatrix}1\\1\end{pmatrix}$.

Now we'll look at negative scale factors. They flip the shape around and onto the other side of the centre of enlargement.



Bottom right was $\begin{pmatrix}2\\1\end{pmatrix}$ from the centre, so will end up $-2 \times \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-4\\-2\end{pmatrix}$

Bottom left: $-2 \times \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-2\\-2\end{pmatrix}$

Top left: $-2 \times \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}-2\\-6\end{pmatrix}$

Step 12) Combined Transformations

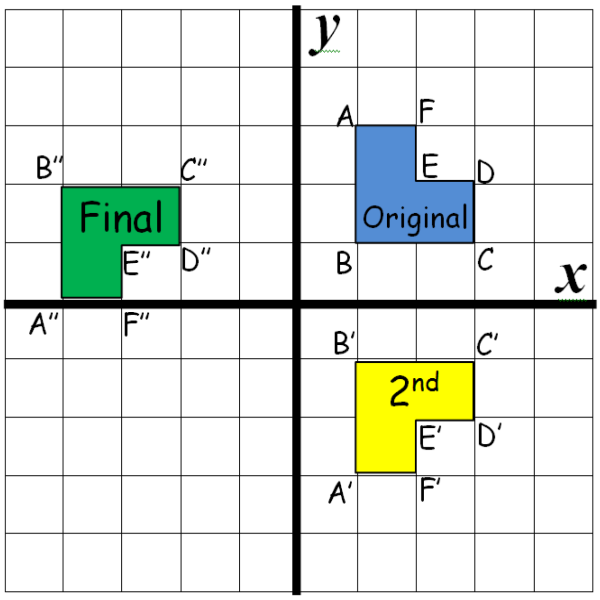
Sometimes we need to be able to combine transformations. After translating, reflecting, rotating or enlarging a shape, we then transform the image as a further step.

Reflect the blue shape ABCDEF in the x axis, colour it and yellow label A'B'C'D'E'F'

This is done below just as we learned in the step on reflection.

Then translate the yellow shape through vector $\begin{pmatrix}-5\\3\end{pmatrix}$, colour green & label A''B''C''D''E''F''

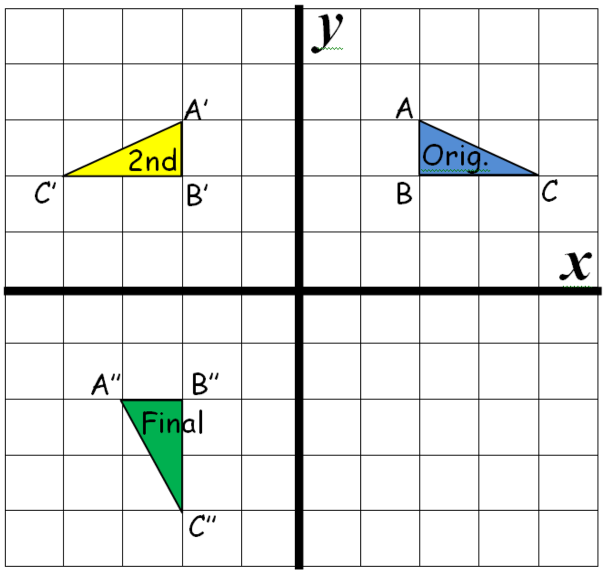
Again this is done below as in the step on translation, but starting with the yellow shape we got AFTER translating the blue shape.



It is useful to note two things here. The shapes have been named and labelled according to their vertices, ie ABCDEF. Secondly transformations of the shape have been called A'B'C'D'E'F'. So each time you transform you can rename the new vertex from A to A' to A'' and even to A''' and so on. This makes it clear that all the A??? vertices are transformations from the original vertex A.

Here is one more example of a combined transformation.

A blue shape ABC was reflected in the y axis below. Then this image was rotated 90^0 anti-clockwise about the origin (0,0) as a centre of rotation. This final shape was labelled A''B''C'' and coloured green. Reverse these translations to find the original blue triangle.



We have to do the opposite translations in the opposite order. So starting with the green triangle we will rotate it 90^0 **clockwise** (anti-clockwise, and clockwise are opposites), labelling this first image A'B'C' and colouring it yellow.

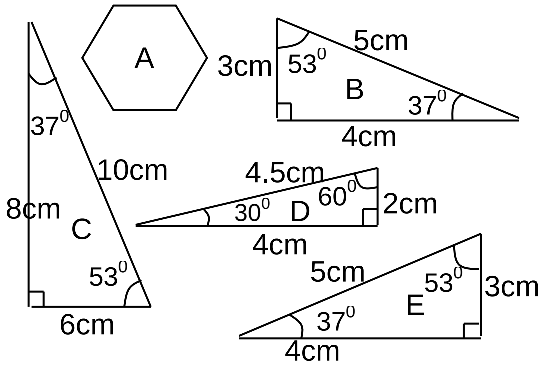
Finally the opposite of a reflection, is reflecting again in the same mirror line (reflection is the only self-inverse transformation). So we reflect the yellow shape A'B'C' in the y-axis and label the shape A''B''C'' and colour it blue. This was our original shape before the first transformation.

Step 13) Similarity & Congruence

Congruent shapes are those which have the same angles and lengths of corresponding sides. They may be rotation (turned around), reflections (mirrored) or translations (moved) relative to one another, but they must be the same size, shape and have the same angles.

Similar shapes have the same angles, but the sides must be the same size relative to one another (ie they are all twice, or three times, or half as big As in the other shape).
Similarity is the relationship you get between shapes that have been enlarged,

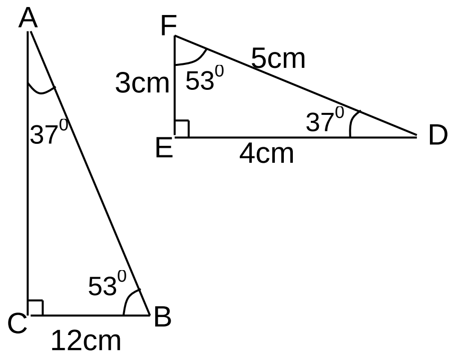
Which shapes are congruent and which shapes are similar?



Congruence: B & E are congruent because they have the same angles (37^0, 53^0 and 90^0)all their sides are the same too, 3cm (between 53^0 and 90^0 angle), 4cm (between 37^0 and 90^0 angle) and 5cm (between 37^0 and 53^0 angle).

B & E are congruent and so similar too (enlargement of scale factor 1). C is similar to them both, it is an enlargement of scale factor x2. Its 6cm (between 53^0 and 90^0 angle) is double B and Es 3cm side, its 8cm (between 37^0 and 90^0 angle) is double B and Es 4cm side, and its 10cm (between 37^0 and 53^0 angle) is double B and Es 5 cm side.

Triangles ABC & DEF are similar. Let's try and find which side in ABC corresponds to the side DE in DEF and how long is it?



DE (4cm) is the side between the 37^0 angle and the 90^0 angle in DEF. AC is between the 37^0 angle and the 90^0 angle in ABC, so AC in ABC corresponds to DE in DEF.

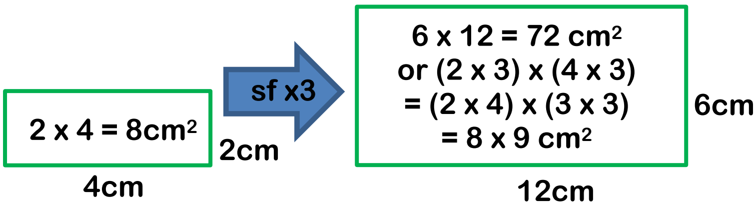
BC corresponds to FE and is 4 times larger, so as they are similar triangles, AC = 4 x DE = 4 x 4 16cm.

Step 14) Length, Area & Volume Scale Factors

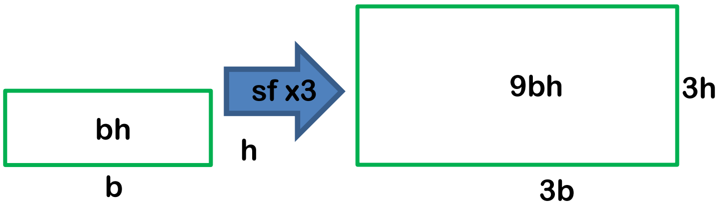
When we enlarge shapes by a given scale factor, we make each side of the shape Sf times longer. This is true for both 2D and 3D shapes.

To learn this in the first place, it is best to get a rectangle and increase it by SFs 2, then 3, then 4, then 5 and see how much the areamultiply by, Then toi do something similar for a cuboid, and investigate its volume as effected by larger and larger scale factors.

Let's enlarge a rectangle by SF 3.



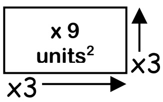
In general we are always multiplying the base and the height by 3 (with scale factor 3)



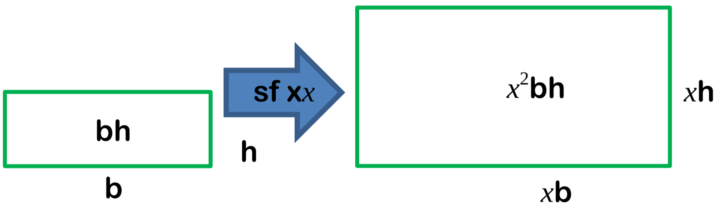
So multiplying the base and the height by 3, means the volume becomes 9 times bigger.

So a scale factor 3, leads to an area increase of 9 times.

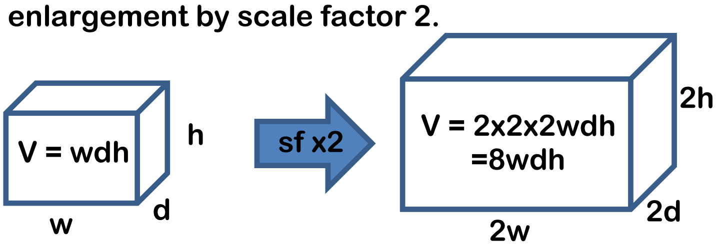
In other words length is x3, so area is x3x3 which is x9



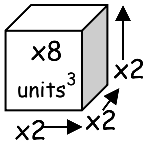
In general a enlargement by scale factor x leads to an area increased x^2 times.



Let's look at the volume of a cube after an enlargement by scale factor 2.

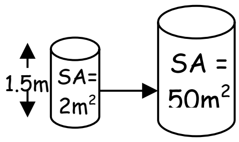


In other words, length is x2, volume is x2x2x2 which is x8



In general when enlarging a 3D shape by scale factor x, the volume gets x^3 times bigger.

These skills can be combined in some very interesting ways. For example lets look at a cylinder of height 1.5m with surface area 2m². If after enlargement it's 50m², how high is the larger cylinder & how many times bigger is its volume.



Well the surface area is 25 times larger. This means the scale factor is $\times\sqrt{25} = \times 5$

So the enlarged height is 1.5 x 5 = 7.5m

Now if the scale factor is x5, the volume will be 5³ times bigger, which is 215 times larger.

Step 15) + - & x with Vectors

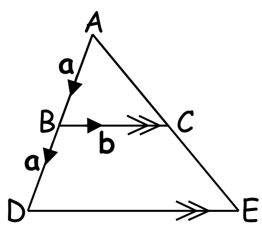
Vectors can be combined by adding, subtracting or multiplying the numerical brackets that represent them.

For example if a = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **& b =** $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$

then $2\boldsymbol{a} = 2 \times \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \times 2 \\ 2 \times 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

And $2a + b = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4-1 \\ 6+5 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \end{pmatrix}$

We can sometimes write a vector as combinations of other vectors.



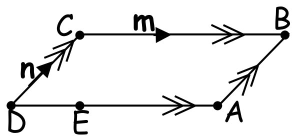
Triangle ADE, is an enlargement sf2 of ABC. Write vector $\overline{BE}$ in terms of **a** & **b**

Similar triangles so $\overline{DE}$ is twice $\overline{BC}$ so $\overline{DE}$= 2

$\overline{BE} = \overline{BD} + \overline{DE}$ **= a + 2b**

$\overline{BE}$ **= a + 2b**

You can prove things with vectors!



The ratio of $\overline{DE}$:$\overline{EA}$ is 1:2.
Show in terms of m & n that $\overline{CE}$ is the same using clockwise and anticlockwise movement.

Anti-Clockwise.

$\overline{CD} = -\boldsymbol{n}$

$\overline{DE} = \frac{1}{3}\overline{DA} = \frac{1}{3}\boldsymbol{m}$

$\overline{CE} = \overline{CD} + \overline{DE}$

$\overline{CE}$ **= -n + ⅓m**

Clockwise

$\overline{CB} = \boldsymbol{m}$

$\overline{BA} = -\boldsymbol{n}$

$\overline{AE} = -\frac{2}{3}\overline{DA} = -\frac{2}{3}\boldsymbol{m}$

$\overline{AE} = \overline{CB} + \overline{BA} + \overline{AE}$

= m -n -⅔m

So
-n + ⅓m = m -n -⅔m

which is true