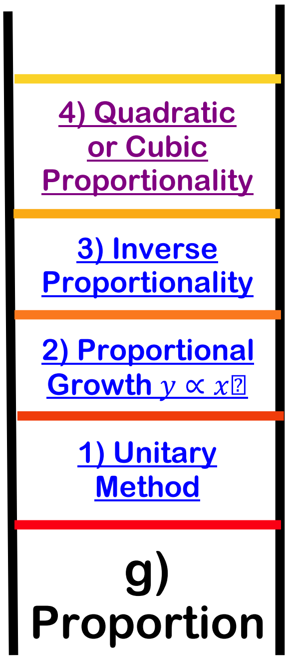
b) Understanding Proportion

1 2 3 4



Step 1) The Unitary Method

Imagine a person who is 1.8m (about 6 feet) tall and only 3cm wide, or perhaps wider than they are tall. They might look unusual, because they are out of proportion. You expect someone will be taller than they are wide, and you don't expect a 6 foot tall human being to be 3cm wide.

Now humans come in all shapes and sizes and the happiest ones learn to see the beauty in everyone. But this idea of proportion can be very precise in many real life situations.

This piece of paper (if you have a printed copy of this book) is A4 size. Its height is about 1.41 times bigger than its width (the exact ratio is $\sqrt{2}$). This ratio, the number of times bigger the height is than the width, is the same for all A-scale paper sizes. So the sides are in proportion to one another in the same way.

So proportion is all about how big something is relative to a whole, or to another thing.

The unitary method lets us move between different quantities or amounts of a given thing. It does this by finding the quantity or amount of 1 unit of this thing.

So if we know that 15kg of potatoes costs £30, we can find how much 1kg should cost by dividing the total cost of 15kg by 15.

1kg = £30
and
1kg = $\frac{15kg}{15}$
so
So 1kg = $\frac{£30}{15}$ = £2

Money has been used as an easy to understand example, but it can be done for almost anything. One limitation for money is that the cost per kg might change if you buy a lot. A market trader who buys 200kg of potatoes will pay less per kg than a customer who buys only ½ a kg.

The mass of a m^3 of concrete doesn't change if you have more m^3s, even though the price per m^3 might reduce. So if we have 10m^3 of concrete and it has a has mass 24 tons, what is the mass of 1m^3 of concrete?

Well
10m^3 = 24 tons
and
1m^3 = $\frac{10\text{m3}}{10}$
so
1m^3 = $\frac{24 \text{ tons}}{10}$ = $2.4\ tons$

This idea that the relative size of the two quantities growing at the same rate is called proportionality. The size of one measure is in proportion to the other if they grow at the same rate. If when doubling one thing (say weight) you also double the other thing (here volume) then the volume and the weight of concrete are said to be proportional. As you will see in the following steps, proportional quantities can easily be put into a simple equation.

Once we can work out the unit price, mass or weight we can work out the price, mass or weight of several of them simply by multiplying up. Let's start with the above example by but try to find the mass of 37m³ of concrete.

$$10m^3 = 24 \text{ tons}$$
so, as before
$$1m^3 = 2.4 \text{ tons}$$

Now if we multiply both sides of this equation by 37, it will tell us the mass of 37m³ of concrete.

$$1m^3 \times 37 = 2.4 \text{ tons} \times 37$$
$$37m^3 = 88.8 \text{ tons}$$

Step 2) Proportional Growth

As we learned in the last step, two things are in proportion when they grow at the same rate. Another way of saying that two things are "in proportion," is saying they are proportional (they have the joint quality of proportionality).

The word portion, as in how much of something you have, is a big clue here. It means the two portions have the same relative size. And if your portion gets bigger, my portion has to get bigger by the same amount for them to stay in pro-portion.

So x and y can be said to be proportional (or in proportion) if doubling x implies that y also doubles. And tripling x means the value of y in that situation will also be 3 times bigger than before. The symbol to say that two things are in proportion is $\propto$. So in this example.

x is proportional to y can be written $x \propto y$
and
y is in proportion to x can be written $y \propto x$

So let's say m and n are proportional, or written in symbols $m \propto n$. If we know that when $m = 3$, then $n = 7$, we can find out other pairs of values, as we know when we double one, we must double the other, and so on.

x10 x3 ÷2

m	3	9	30	15
n	7	21	70	35

x3 ÷2 x10

So the fact that $m \propto n$ and if $m = 3$, then $n = 7$

we also know by tripling both that

when $m = 3 \times 3 = 9$, then $n = 7 \times 3 = 21$

and by multiplying both by 10 that

when $m = 3 \times 10 = 30$, then $n = 7 \times 10 = 70$

We can start from any of these related pairs and create another pair by times or dividing by the same number.

So as when $m = 30$, then $n = 70$, if I divide both by 2 then we know that

When $m = \frac{30}{2} = 15$, then $n = \frac{70}{2} = 35$.

For a given proportional relationship, as long as we know 1 pair of related values, then we can find the pair with any other value.

Let's say that $y \propto x$ and that when $x = 6$ then $y = 9$

Could we find what y is when $x = 3$ or what x is when $y = 36$? Let's write this information in a table.

x	*3*	*6*	*?*
y	*?*	*9*	*36*

We need to find what we divide or times $x = 6$ by or $y = 9$ by to find the other values of these variables, and do the same for the paired variable.

÷2

x	*3*	*6*	*?*
y	*?*	*9*	*36*

x4

When $y = 36,\ x = 6 \times 4 = 24$

When $x = 3,\ y = \frac{9}{2} = 4.5$

So

x	*3*	*6*	*24*
y	4.5	9	36

Is the completed table

Proportional relationships can amazingly be summed up in a very simple formula.

Remind yourself of the inverse relationship between x and ÷, reading step 7 of ladder a on +. -, x and ÷.

Let's start with a proportional relationship $y \propto x$ with starting values $x = 1$ and $y = 3$. Firstly we'll make a table with a few more pairs of values.

x	1	2	3	4	5
y	3	6	9	12	15

Let's look at the value for each pair of $\frac{y}{x}$

x	1	2	3	4	5
y	3	6	9	12	15
$\frac{y}{x}$	3	3	3	3	3

This is interesting, $\frac{y}{x}$ is always a constant number, here 3.

That means that $\frac{y}{x} = 3$.

Rearranging, this equation can be written more beautifully as $y = 3x$

In general this is true and $\frac{y}{x} = k$ where k is a constant number, fixed within that particular proportional relationship. And this relationship can be written as equation $y = kx$. So our job is simply to find out what k is.

This equation can then be used to find missing values very easily. If $y = 3x$, and we want to know the value of y when $x = 100$ we simply substitute in to our equation $y = 3 \times 100 = 300$

Let's look back at our previous example then, where $y \propto x$ and the following were related pairs of values.

x	*3*	*6*	*24*
y	*4.5*	*9*	*36*

Let's find an equation to represent this proportional relationship, and use it to find y when $x = 12.7$

Because $y \propto x$ we know that $y = kx$ and k can be found with any x, y pair using $k = \frac{y}{x}$ (this is just a rearrangement of the equation $y = kx$).

$$y = kx$$

$$9 = k \times 6$$

$$k = \frac{9}{6} = 1.5$$

$$y = 1.5x$$

It is important to know that we can use any pair of values with $k = \frac{y}{x}$ to find the magic constant k. Let's use pair $x = 24$, $y = 36$

$$y = kx$$

$$36 = k \times 24$$

$$k = \frac{36}{24} = 1.5$$

$$y = 1.5x$$

In the same way, you can derive the equation $y = 1.5x$ using any pair values found within this proportional relationship.

Finally we will find the related y -value when $x = 12.7$

$$y = 1.5x$$

$$y = 1.5 \times 12.7$$

So $y = 19.05$, when $x = 12.7$

Step 3) Inverse Proportionality

Inverse proportionality comes from the fact that x and ÷ are inverses. With inverse proportionality, if you multiply x by a number, you find the related y pairing by dividing by the same number. So if I have and x and y pairing I can find another pair by doubling the x-value and halving the y-value.

The phrase y is inversely proportional to x is symbolised by writing $y \propto \frac{1}{x}$

Let's try and fill in the rest of this table where $y \propto \frac{1}{x}$

x	2	4	1
y	10		

Now in the first missing y-value, we have mu;tiplies the x-values by 2 (2 x 2 = 4+ so we divide the y-value of 4 by 2 = 20.

Then from the x-value 4 to $x = 1$ we have divided by 4, so we multiply the (new) y-value of 5 by 4, which gives $y = 20$.

x2 ÷4

x	2	4	1
y	10	5	20

so ÷2 so x4

Again there is an easy way to make a formula for inverse proportionality. Here let's investigate xy.

x	2	4	1
y	10	5	20
xy	20	20	20

It looks like xy is always the same, here 20. This is true for all inversely proportional relationships: $xy = k$ where k is a constant. So here $xy = 20$.

This can be rearranged into the beautiful equation $y = \frac{20}{x}$ which is nicely in the same form as the general way we symbolise inverse proportionality $y \propto \frac{1}{x}$.

Again we can use this formula to find lots of related pairs of x-values and y-values.

Let's look at an example where $y \propto \frac{1}{x}$ (y is inversely proportional to x) , where $y = 24$ is when $x = 2$ =creating a formula and using it to find the value of y when $x = 7.5$

$$y = \frac{k}{x}$$

$$k = xy$$

$$k = 2 \times 24 = 48$$

so

$$y = \frac{48}{x}$$

Now when $x = 7.5$

$$y = \frac{48}{7.5} = 6.4$$

Step 4) Quadratic or Cubic Proportionality

We have looked at what happens when y is proportional to x written $y \propto x$, and when y is inversely proportional to x, written $y \propto \frac{1}{x}$

In quadratic and cubic proportionality, we find that y is proportional to x^2 or x^3 or in the case of inverse proportionality to $\frac{1}{x^2}$ or $\frac{1}{x^3}$

We can work with these relationships in just the same way as before, except we find the following in each case.

Type	Written	Form of equatio n	How to find k
Direct Quadratic	$y \propto x^2$	$y = kx^2$	$k = \frac{y}{x^2}$
Direct Cubic	$y \propto x^3$	$y = kx^3$	$k = \frac{y}{x^3}$
Inverse Quadratic	$y \propto \frac{1}{x^2}$	$y = \frac{k}{x^2}$	$k = yx^2$
Inverse Cubic	$y \propto \frac{1}{x^3}$	$y = \frac{k}{x^3}$	$k = yx^3$

We'll investigate the first one more thoroughly to give an understanding of how this works, and then do some shorter examples.

Where you multiplied or divided x and y by the same number with direct proportionality, and used the same number but switched times for divide with inverse proportionality, here you have to square (or cube) the y-values to keep the things in proportion.

Here is an example of some x, y pairs where the relationship is direct proportionality between y and x^2

÷5 x2

x	1	5	10
x^2	1	25	100
y	2	50	200

÷5² = ÷25 x2² = x4

Let's find a formula for this relationship and use it to find y when $x = 1.5$

Now $y \propto x^2$ which means that $y = kx^2$ and we can find k using $k = \frac{y}{x^2}$

We can use any pair of values to find k

$$k = \frac{y}{x^2} = \frac{2}{1^2} = \frac{50}{5^2} = \frac{200}{10^2} = 2$$

So $y = 2x^2$ and hence when $x = 1.5$

$$y = 2 \times 1.5^2 = 4.5$$

Let's look at cubic proportionality. What is the relationship between these two variables?

x	1	2	4
y	2	16	128

Let's look at how we get from $x = 2$ to the other x-values.

÷2 x2

x	1	2	4
y	2	16	128

÷2³ = ÷8 x2³ = x8

Whatever we times or dived x, we have to cube what we times or divide y by. So this is cubic proportionality.

Let's find a formula, and use it to find y when $x = 3.5$

$y \propto x^3$ which means that $y \propto x^3$

and we can find k using $k = \frac{y}{x^3}$

So $k = \frac{16}{2^3} = 2$

Hence $y = 2x^3$

So when $x = 3.5$

$y = 2 \times 3.5^3 = 85.75$

Inverse proportionality works similarly, but where we multiply one variable we divide the other.

Let's say that $y \propto \frac{1}{x^3}$

and that when $x = 2,\ y = 25$

Let's find a formula to relate x and y

$y = \frac{k}{x^2}$ and so $k = yx^2$

$$k = 25 \times 2^2 = 100$$

So $y = \frac{100}{x^2}$

Let's use this equation to fill in the missing details in this table.

x	?	2	5
y	100	25	?

$y = \frac{100}{x^2}$ so when $y = 100$

$$100 = \frac{100}{x^2}$$

so $x^2 = \frac{100}{100} = 1$

so $x = 1$

Similarly when $x = 5$

$$y = \frac{100}{5^2} = 4$$

And the completed table is

x	**1**	**2**	**5**
y	**100**	**25**	**4**