u) Understanding Equations

u) Equations

Step 1) True Equations

You have been using equations in maths for a long time.

For example $2 \times 3 = 6$
and $2 \times 3 = 7$
are both equations. One of them is true and one of them is false. It is true to say that $2 \times 3 = 6$ because if you got two lots of 3, you have 3 + 3 which makes 6 (not 7). So $2 \times 3 = 7$ is false.

Here we are going to look at how equations involving variable numbers (represented as letters) can be false or true.

In ladder s we saw that there are some very special types of equations where both expressions (on each side of the equation) are always the same whatever the value of the variables.

for example $2x + 3x \equiv 5x$

This is always true whatever value of x we look at.

But these special $\equiv$ equations are not the type we usually work with.

There are an infinite number of equations that are only true for certain values of the variable.

$$2x + 1 = 7$$

This is true for some values of x and false for others.

	LHS $2x+1$	RHS 7	True/False
When $x = 1$	3	7	False as $3 \neq 7$
When $x = 2$	5	7	False as $5 \neq 7$
When $x = 3$	7	7	True as $7 = 7$
When $x = 4$	9	7	False as $9 \neq 7$
When $x = 5$	11	7	False as $11 \neq 7$

You can see that the equation $2x + 1 = 7$ is true when $x = 3$, and is false for the other four values of x we tried. In fact this equation is only true when $x = 3$.

We have a short word for this true value, we call a value of the variable for which the equation is true a solution. So here $x = 3$ is a solution of the equation $2x + 1 = 7$ (because this equation is true when $x = 3$)

We will later discover that linear equations have a maximum of 1 solution, quadratics have a maximum of 2 solutions, cubics a maximum of 3 solutions and so on.

Let's look at a quadratic equation now.

$x^2 = 25$

This equation is true when $x = 5$ and also when $x = -5$. Now we told you just above that quadratics have a maximum of 2 solutions, but let's try a few more just to take a look out of interest.

	LHS	RHS	True/False
	x^2	25	
When $x = 3$	9	25	False as $9 \neq 25$
When $x = -3$	9	25	False as $9 \neq 25$
When $x = 4$	16	25	False as $16 \neq 25$
When $x = -4$	16	25	False as $16 \neq 25$
When $x = 5$	25	25	True as $25 = 25$
When $x = -5$	25	25	True as $25 = 25$
When $x = 6$	36	25	False as $36 \neq 25$
When $x = -6$	36	25	False as $36 \neq 25$
When $x = 7$	49	25	False as $49 \neq 25$
When $x = -7$	49	25	False as $49 \neq 25$

To summarise, an equation is either true or false when we substitute in a particular value of the variable. If it is true, then this value is called a solution for that equation.

It turns out that these "solutions" for equations that represent or express real life situations, very often help us to solve the actual real life problem that those equations express. For this reason mathematicians spend a lot of time learning how to find the solution(s) to an equation, in other words finding the value(s) that make an equation true.

You can create a very complicated true equation by starting with the solution. Let's say you want to make a true equation whose solution is $x = 3$

We simply do things (like +1, or square, or x7) to both sides of the equation. If the LHS and RHS (Left Hand Side and Right Hand Side) of an equation are the same, then if we do the same thing to each of them, the LHS and the RHS of the new equation we create will be the same (in other words the new equation will be true if the one we started with was true).

Equation	Do	Notes
$x = 3$	(x2)	LHS $x \underline{\times 2} \equiv 2x$ RHS $3 \underline{\times 2} \equiv 6$
$2x = 6$	(+7)	LHS $2x \underline{+7} \equiv 2x + 7$ RHS $6 \underline{+7} \equiv 13$
$2x + 7 = 13$		

All the equations are true when $x = 3$, this includes $2x = 6$ and $2x + 7 = 13$

Let's try something a little more complex, but still starting with $x = 3$.

Equation	Do	Notes
$x = 3$	(square)	LHS $x \times x \equiv x^2$ RHS $3^2 \equiv 9$
$x^2 = 9$	(x2)	LHS $x^2 \underline{\times 2} \equiv 2x^2$ RHS $9 \underline{\times 2} \equiv 18$
$2x^2 = 18$	(-1)	LHS $2x^2 \underline{-1} \equiv 2x^2 - 1$ RHS $18 \underline{-1} \equiv 17$
$2x^2 - 1 = 17$	(+7x)	LHS $2x^2 - 1 \underline{+7x} \equiv 2x^2 + 7x - 1$ RHS $17 \underline{+7x} \equiv 7x + 17$
$2x^2 + 7x - 1 = 7x + 17$		

This very complex equation $2x^2 + 7x - 1 = 7x + 17$ is still true when $x = 3$

Step 2) Solving Equations by Trial & Improvement with Whole Nos

Solving an equation is the process of trying to find the solutions of that equation, or the values of the variables that make the equation true.

Many simple equations can be solved simply by spotting what the solution is.

$x + 2 = 5$

If you have a good sense of number you might be able to spot that as $3 + 2 = 5$ so $x = 3$ is the solution to our equation.

If you can't spot it you could try trial and improvement. This is where you try a value of x and, if it didn't make the equation true, you then decide from the substitution whether to next try a bigger or smaller value of x next.

	LHS	RHS	True/False
	$x + 2$	5	
Try $x = 1$	3	5	False, try bigger
Try $x = 2$	4	5	False, try bigger
Try $x = 3$	5	5	TRUE

Generally if our substitution is too small we try a bigger value of x, though sometimes (like when there are negative numbers involved) this can have the opposite effect.

Let's try and find the solution for another equation

$7x = 56$

	LHS	RHS	True/False
	$7x$	56	
Try $x = 5$	35	56	False, try

			bigger
Try $x = 6$	42	56	False, try much bigger
Try $x = 9$	63	56	False, try smaller
Try $x = 8$	56	56	TRUE

Now if you know your times tables particularly well you might have spotted that $7 \times 8 = 56$ as soon as you saw the equation, and from this known that $x = 8$ was the solution to this equation.... but what about when the equation has 2 terms, or quadratic terms, or the answer is a decimal, it becomes harder and harder to "spot"solutions.

Let's try another equation

$$7x - 6 = 57$$

	LHS $7x - 6$	RHS 57	True/False
Try $x = 5$	29	57	False, try much bigger
Try $x = 8$	50	57	False, try a bit bigger
Try $x = 9$	57	57	TRUE

So the value $x = 9$ makes the equation $7x - 6 = 57$ true, it is the solution to this equation.

Now $7 \times 2.63 - 7.58 = 10.83$

Which means that the solution to the equation
$7x - 7.58 = 10.83$
is $x = 2.63$

We'll see later on in step 8 that we can solve many equations in this way, but when we look at decimals (or quadratics or cubics that have multiple solutions) things are going to get very complicated. We need a method that allows us to solve equations without having to try loads of different values. A method that could quickly and easily solve an equation like
$7x - 7.58 = 10.83$

Step 3) Solving Equations with Inverses

To solve an equation you just take the inverse.

This rather fancy word inverse just means the opposite.

+3 and -3 are inverses (or opposites)

x7 and ÷7 are inverses

-1.5 and +1.5 are inverses

÷3.1 and x3.1 are inverses.

So if we look at the equation
$2x + 7 = 13$ we created in step 1...

Equation	Do	Notes
$x = 3$	(x2)	LHS $x \times 2 \equiv 2x$ RHS $3 \times 2 \equiv 6$
$2x = 6$	(+7)	LHS $2x + 7 \equiv 2x + 7$ RHS $6 + 7 \equiv 13$
$2x + 7 = 13$		

We simply do the inverse operations in the opposite order.

So to create $2x + 7 = 13$, we started with the solution $x = 3$, then x2, then +7. So we do the inverses in the opposite order, first -7, then ÷2.

Equation	Do	Notes
$2x + 7 = 13$	(-7)	LHS $2x + 7 - 7 \equiv 2x$ RHS $13 - 7 \equiv 6$
$2x = 6$	(÷2)	LHS $2x \div 2 \equiv x$ RHS $6 \div 2 \equiv 3$
$x = 3$		

We didn't actually need to know that the solution was $x = 3$ to do this, we can solve the equation starting from the equation and getting to the solution. Without the notes we can simply write...

$2x + 7 = 13$ (-7)
$2x = 6$ (÷2)
$x = 3$

Looking at the order we substitute into $2x + 7 = 13$ we can establish the "forwards" order is x2, +7, so the inverse order is -7, then ÷2!

Now for this equation $2x + 7 = 13$ it might be easier to just spot the answer, trying $x = 1$, $x = 2$, $x = 3$. but what about for $7x - 7.58 = 10.83$

Well the forwards order is x7, then -7.58, so the inverse order is +7.58 then ÷7.

Equation	Do	Notes
$7x - 7.58 = 10.83$	(+7.58)	LHS $7x - 7.58 + 7.58 \equiv 7x$ RHS $10.83 + 7.58 \equiv 18.41$
$7x = 18.41$	(÷7)	LHS $7x \div 7 \equiv x$ RHS $18.41 \div 7 \equiv 2.63$
$x = 2.63$		

Without the notes we can just write...
$7x - 7.58 = 10.83$ (+7.58)
$7x = 18.41$ (÷7)
$x = 2.63$

Step 4) Solving Eqns with Variables Both Sides

$$5x - 3 = 2x + 15$$

We can't do this exactly as we did the others because there is no specific order in which we would substitute to find our forwards order.

The problem is that we have two variable terms, one on each side of the equation. If we were on the same side we'd just collect them together like in step 4 of the expressions ladder on collecting like terms, but they are on opposite sides! The solution is that either of these terms could be removed by doing the inverse operation, and that would mean that the inverse term appeared on the other side. We are effectively moving the term from one side to the other. So we would either start by inversing 2x, which as it is positive means we inverse by doing -2x. Or we could inverse the 5x, which would be -5x. Let's try both.

Equation	Do	Notes
$5x - 3 = 2x + 15$	$(-2x)$	LHS $5x - 3 - 2x \equiv 3x - 3$ RHS $2x + 15 - 2x \equiv 15$
$3x - 3 = 15$		
At this point we have a standard linear equation, with "forwards" order x3, -3, so the inverse is +3, ÷3		
$3x - 3 = 15$	$(+3)$	LHS $3x - 3 + 3 \equiv 3x$ RHS $15 + 3 \equiv 18$
$3x = 18$	$(\div 3)$	LHS $3x \div 3 \equiv x$ RHS $18 \div 3 \equiv 6$
$x = 6$		

Or lets try cancelling the 5x, by starting with a -5x step.

Equation	Do	Notes
$5x - 3 = 2x + 15$	$(-5x)$	LHS $5x - 3 - 5x \equiv -3$ RHS $2x + 15 - 5x$ $\equiv -3x + 15$
$-3 = -3x + 15$		
At this point we have a standard linear equation, with "forwards" order x-3, +15, so the inverse is -15, ÷-3		
$-3 = -3x + 15$	(-15)	LHS $-3 - 15 = -18$ RHS $-3x + 15 - 15 \equiv -3x$
$-18 = -3x$	$(\div -3)$	LHS $-18 \div -3 \equiv 6$ RHS $-3x \div -3 \equiv x$
$6 = x$	which means the same as	
$x = 6$		

The first of these methods minimises the use of negative numbers, so one of the ways is easier, though both work. Without the notes, using the "easier method" it would look like this...

$$5x - 3 = 2x + 15 \qquad (-2x)$$

$$x - 3 = 15 \qquad (+3)$$

$$3x = 18 \qquad (\div 3)$$

$$x = 6$$

Step 5) Formulae

A formula is a special type of equation that we use to calculate something useful, we call this useful thing the subject. Formulae usually have more than one variable.

For example the formula for the area of a rectangle is given as

Where A = the area of the rectangle
b = the length of the base of the rectangle
h = the height of the rectangle

A = b x h

Here A is the subject.

So for some particular rectangles, when we know the base and the height we can work out the Area.

The formula for calculating a missing angle c in a triangle is c = 180 – a – b where a & b are the other two angles in the triangle. Here c is the subject of the formula.

We have many formulae within maths for quickly calculating many different things, and many more still within the sciences.

Let's look out a formula without a context.

m = 4y – 3p

To find m (the subject) in a given situation we need to know y and p, and we must take 3p away from 4y. Every time we know the value of p and y in a given situation we can then easily calculate m using this formula.

Let's take 3 different situations where we know y and p and find m.

		4y	3p	m
Situation 1	y=2 p=1	8	3	5
Situation 2	y=1 p=2	4	6	-2
Situation 3	y=5 p=-2	20	-6	26

We can also use the same formula given the value of the subject and some other variable(s). We can then form an equation to solve to find the value of an unknown variable.

Let's take another situation where the formula m = 4y – 3p applies.
If we know that m = 25 and p = 1
Then we can form equation...

25 = 4y – 3

p = 1
-3p = -3

m = 25

We then solve the equation 25 = 4y – 3

$$\begin{aligned} 25 &= 4y - 3 && (+3) \\ 28 &= 4y && (\div 4) \\ 7 &= y && \text{(swap LHS \& RHS)} \\ y &= 7 \end{aligned}$$

We will see in the next step, that if we had lots of values of m and p, we could rearrange our equation to make y the subject.

6) Changing the Subject of a Formula

At the end of the last step we substituted in a value of m and p to the equation m = 4y – 3p

We then solved the equation formed to find the value of y needed to solve this equation. But if we had lots of values of m and p, we could rearrange the formula to make y the subject.

The forwards order for y is x4, then -3p, so we need to +3p then ÷4

$$\begin{aligned} m &= 4y - 3p && \textbf{(+3p)} \\ m + 3p &= 4y && \textbf{(÷4)} \\ \frac{m+3p}{4} &= y && \textbf{Swap LHS \& RHS} \\ y &= \frac{m+3p}{4} \end{aligned}$$

Let's look at example $2x^2 - 3m = 7$ which is quadratic with x. Forwards from x we square, then x2, then -3m, so the inverse is +3m, ÷2, then √ square root,

$$\begin{aligned} 2x^2 - 3m &= 7 && \textbf{(+3m)} \\ 2x^2 &= 3m + 7 && \textbf{(÷2)} \\ x^2 &= \frac{3m+7}{2} && \surd \\ x &= \sqrt{\frac{3m+7}{2}} \end{aligned}$$

Sometimes we have the new subject variable in more than one term, for this we need to collect them together and factorise.

$$\begin{aligned} px - 7k &= 2vc + 5x && \textbf{(+7k)} \\ px &= 2vc + 5x + 7k && \textbf{(-5x)} \\ px - 5x &= 2vc + 7k && \textbf{(Factorise)} \\ x(p-5) &= 2vc + 7k && \textbf{(÷(p-5))} \\ x &= \frac{2vc+7k}{p-5} \end{aligned}$$

Step 7) Inequalities

We have learned a lot about things that are equal. We have seen that some things are always equal like $2 \times 3 = 6$ and $2x + 5x = 7x$. Here we can use the triple equals sign to represent that they are always equal $2 \times 3 \equiv 6$ and $2x + 5x \equiv 7x$, whatever the value of x.

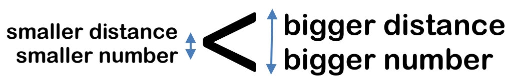
We have learned that other things are sometimes equal. $5x + 7 = 22$ only and exactly when $x = 3$. Any other value of x leaves this equation untrue.

We also know that some things are not equal. For example 3 is not equal to 7 – 1. We can either say that the equation $7 - 1 = 3$ is false, or we can use the not equal symbol $7 - 1 \neq 3$.

There is another type of relationship between numbers that helps us talk about them. Not only if they are equal or not, but in the situation where they are not equal, which one is bigger.

We use the symbols > and < to notate numbers that are not equal when we know which one is bigger.

We always put the smaller number at the point end of the symbol, and the large end at the double end of the symbol. One nice way to remember this is that there's a larger distance at the double end.



You can also think of the inequality (literally meaning not equal, or in-equal) symbol as a set of teeth saying the little number is eats the bigger number.

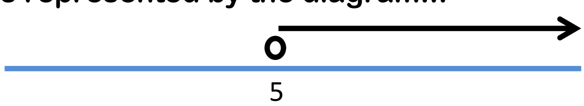
So we can say that seven is less than 12 simply as $7 < 12$, and $100 < 3{,}476$, $345 > 9$, $7 > -3$, and $-23 < -5$.

However this has a greater use within algebra, for working with variable numbers.

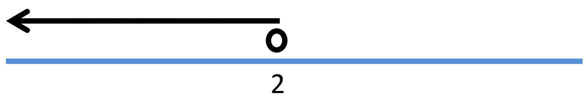
If we know that $x > 5$ and x is a whole number than x could be 6, 7, 8, 9, 10...

If x could be a decimal then it could also be 5.1, 5.00001, 5.9, 8.7 and many other possibilities.

It is easier to understand the numbered list 6, 7, 8, 9, 10... than the decimal version, so when we are talking about all decimals bigger or smaller than a certain number we can use an inequality diagram. $x > 5$ is represented by the diagram...



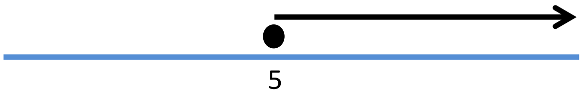
And $x < 2$ by diagram



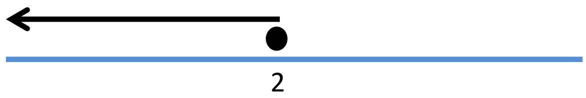
With algebraic inequalities, it can sometimes be useful to say that a number is greater than or equal to. That means it might be bigger than, or equal to the other number but cannot be smaller than it.

We use the symbol $x > 5$ to not include 5 as a possibility, and $x \geq 5$ (read x is greater than or equal to 5) to show that it is. So if they were whole numbers $x > 5$ means $x = 6, 7, 8, 9, 10 \dots$
and $x \geq 5$ means $x = 5, 6, 7, 8, 9 \dots$

On a number line including all the decimals (not just whole numbers) we represent this by filling in the dot representing the 5. So $x \geq 5$ is represented by



and $x \leq 2$ by diagram



We can solve inequalities just like we solve equations.

Earlier we solved $2x + 7 = 13$ to find the solution was $x = 3$

$$2x + 7 = 13 \quad \text{(-7)}$$
$$2x = 6 \quad (\div 2)$$
$$x = 3$$

But if we make x a little bigger then they value of $2x + 7$ will become a little bigger, in other words a little bigger than 13

So if $2x + 7 > 13$ we know that $x > 3$
Similarly if $2x + 7 < 13$ we know that $x < 3$

We can work with inequalities as with equations (with one exception below) because if we double both sides, the bigger side will still be bigger, and if we +3 to both sides the same is still true and so on…

So we can solve $2x + 7 < 13$ by doing

$$2x + 7 < 13 \quad \text{(-7)}$$
$$2x < 6 \quad (\div 2)$$
$$x < 3$$

And So we can solve $2x + 7 < 13$ by doing

$$2x + 7 < 13 \quad \text{(-7)}$$
$$2x < 6 \quad (\div 2)$$
$$x < 3$$

The exception then is if we x or ÷ by a negative number. Because for example $-2 < -1$ but $1 < 2$ the inequality actually flips directions if we x or ÷ by a negative number. We can solve most inequalities simply by working around this.

Similarly squaring both sides can cause a problem if one side of an inequality was negative, as, for example, $(-2)^2 = 2^2 = 4$.

Step 8) Solving Equations by Trial & Improvement with Decimals

$x^3 + 3x^2 = 26$ can't be solved simply by rearranging the equation, but we can keep making better and better guesses to get a good idea of the solution.

Here we'll try and first narrow down the whole number solution, then to 1dp, then to 2dp.

Try	LHS $x^3 + 3x^2$	RHS 26	LHS too big/small	Means
$x = 2$	20	26	small	$x > 2$
$x = 3$	50	26	big	$x < 3$
So $2 < x < 3$ now test midpoint				
$x = 2.5$	34.375	26	big	$x < 2.5$
So $x = 2$ (nr whole no.)				
$x = 2.1$	22.491	26	small	$x > 2.1$
$x = 2.2$	25.168	26	small	$x > 2.2$
$x = 2.3$	28.037	26	big	$x < 2.3$
So $2.3 < x < 3$ now test midpoint				
$x = 2.25$	26.578	26	big	$x < 2.25$
$x = 2.2\ (1dp)$				
$x = 2.24$	26.292	26	big	$x < 2.24$
$x = 2.23$	26.008	26	big	$x < 2.23$
$x = 2.22$	25.726	26	small	$x > 2.22$
So $2.22 < x < 2.23$, now test midpoint				
$x = 2.225$	25.867	26	small	$x = 2.225$
So $x = 2.23\ (2dp)$				

So bit by bit, we can move to a more and more accurate answer, narrowing it down to two consecutive 1dp, 2dp, 3dp possibilities and testing the midpoint.

Step 9) Solving Linear Eqns with Fractions

This step is about understanding that fractions are just a way of dividing. So $\frac{1}{2}x \equiv \frac{x}{2}$ which is $x \div 2$, and the inverse of ÷2 is x2.

Equation	Do
$\frac{1}{2}x + 3 = 8$	$\frac{1}{2}x \equiv \frac{x}{2}$
$\frac{x}{2} + 3 = 8$	(-3)
$\frac{x}{2} = 5$	(x2)
$x = 10$	

Similarly $\frac{3}{4}x \equiv \frac{3x}{4}$ which forwards from x is x3, ÷4, so the inverse is to x4, then ÷3.

Equation	Do
$\frac{3}{4}x + 2 = 11$	$\frac{3}{4}x \equiv \frac{3x}{4}$
$\frac{3x}{4} + 2 = 11$	(-2)
$\frac{3x}{4} = 9$	(x4)
$3x = 36$	(÷3)
$x = 12$	

10) Solving Simultaneous Equations

When you solve an equation you are finding the value(s) of the variable that makes the equation true. Solving simultaneous equations is finding the values of the variable that make a pair of (or more) equations both true at the same time.

For example $3x = 15$ and $10x = 50$ are both true, that is they are true simultaneously, when $x = 5$. But this is also the solution for each of those equations without each other. For equations with more than one variable, you can often find specific solutions that make them true at the same time.

The basis of solving simultaneous equations is doing operations until one variable term in each equation is either the same (and then subtracting them) or the negatives of one another (and then adding them).

Let's start with an example where the y term in each equation is already the same.

Do	Equation	Name
	$2x + 3y = 8$	a
	$5x + 3y = 11$	b
b – a	$3x = 3$	c
- two equations means to do RHS - RHS to form the new RHS, and LHS - LHS to make the new LHS		
c÷3	$x = 1$	
sub $x = 1$ into a	$2 + 3y = 8$	
(-2)	$3y = 6$	
(÷3)	$y = 2$	

So the solution to these simultaneous equations is $x = 1$ and $y = 2$. This pair of values make both equations (a and b) true simultaneously.

$2x + 3y = 8$ gives $2 + 6 = 8$ which is true and
$5x + 3y = 11$ gives $5 + 6 = 11$ which is also true!

Let's try and example where the two variables are the negatives of one another, here we add the simultaneous equations.

Do	Equation	Name
	$4x + 5y = 23$	a
	$2x - 5y = -11$	b
a + b	$6x = 12$	
+ two equations means to do RHS + RHS to form the new RHS, and LHS + LHS to make the new LHS		
(÷6)	$x = 2$	
sub $x = 2$ into a	$8 + 5y = 23$	
(-8)	$5y = 15$	
(÷3)	$y = 3$	

So here the solutions are $x = 2$ and $y = 3$

Again it is important to understand that this pair of values are a solution because they make both equations true simultaneously (at the same time).

$4x + 5y = 23$ gives $8 + 15 = 23$ and
$2x - 5y = -11$ gives $4 - 15 = -11$, both are true!

It might be that the equations can't just be added or subtracted to cancel a term… then we look for a term that can be multiplied up to the "size" of another term.

Do	Equation	Name
	$2x + 2y = 8$	a
	$6x + 3y = 18$	b
3a	$6x + 6y = 24$	c
c – b	$3y = 6$	
(÷3)	$y = 2$	
sub $y = 2$ into a	$2x + 4 = 8$	
(-4)	$2x = 4$	
(÷2)	$x = 2$	

So here $x = 2$ and $y = 2$. Mentally check if they make equations a and b true… They do!

Sometimes we have to adjust both equations to get two equations where the terms will cancel by + or – the equations.

Do	Equation	Name
	$2x + 5y = 8$	a
	$4y - 3x = 11$	b
ax3	$6x + 15y = 24$	c
bx2	$8y - 6x = 22$	d
c + d	$23y = 46$	
(÷23)	$y = 2$	
sub $y = 2$ into a	$2x + 10 = 8$	
(-10)	$2x = -2$	
(÷2)	$x = -1$	

So the solution is $x = -1$ and $y = 2$.

Finally, sometimes one of the equations is a quadratic. Then we can substitute the linear one, into the other, and solve it.

Do	Equation	Name
	$x^2 + y^2 = 25$	a
	$y = 2x - 2$	b
sub b into a	$x^2 + (2x - 2)^2 = 25$	
Multiply brackets	$x^2 + (4x^2 - 8x + 4) = 25$	
Simplify LHS	$5x^2 - 8x + 4 = 25$	
(-25)	$5x^2 - 8x - 21 = 0$	
You will be learning to solve quadratics in the next two steps, but here goes...		
Factorise LHS (see next two steps)	$(5x + 7)(x - 3) = 0$	
One bracket is 0	$x - 3 = 0$	c
(+3)	$x = 3$	
Or the other bracket is 0	or $5x + 7 = 0$	d
(-7)	$5x = -7$	
(÷5)	$x = -\frac{7}{5}$	

So $x = -\frac{7}{5}$ and $x = 3$ are the two x solutions. We can sub them into equation b to find the matching y solutions.

When $x = 3$, then $y = 2x - 2 = 6 - 2 = 4$

So $x = 3$ and $y = 4$ is one solution pair.

So $x = -\frac{7}{5}$, then $y = 2x - 2 = -\frac{14}{5} - \frac{10}{5} = -\frac{24}{5}$
And $x = -\frac{7}{5}$, and $y = -\frac{24}{5}$ is the other.

11) Solving Quadratic Equations when a=1

We have already learnt in the brackets ladder how to multiply out two linear brackets to make a quadratic, and when possible, how to factorise a quadratic to make two multiplied linear bracket factors.

Multiplying out we learn that...

$$(x + 3)(x + 2) \equiv x^2 + 5x + 6$$

And factorising we learn that...

$$x^2 + 5x + 6 \equiv (x + 3)(x + 2)$$

In other words the expression $x^2 + 5x + 6$ and the expression $(x + 3)(x + 2)$ are effectively the same thing, and so can be swapped for each other (in brackets) in any situation.

Using factorisation a quadratic equation becomes possible to solve! Let's try and solve $x^2 + 5x + 6 = 0$
Because of what we learned above the LHS can be swapped for $(x + 3)(x + 2)$

So solving $x^2 + 5x + 6 = 0$ will give the same solutions as solving $(x + 3)(x + 2) = 0$
or indeed solving $(x + 2)(x + 3) = 0$

This is where there is one further thing to understand. We have now two brackets multiplied together that make 0. When the product (result of multiplying) two numbers is 0, one of the numbers must be 0.

Let's look at two numbers a and b, and say that when we multiply them they make 0.

Here are some examples that make 0 when multiplied.

a	b	ab
0	0	0
0	7	0
9	0	0
0	23.7	0
0	-19	0
5	0	0
3π	0	0
0	7e	0

Here are some examples of a and b that don't make 0 when multiplied.

a	b	ab
1	2	3
3	7	21
9	4	36
2	23.7	47.4
3	-19	-57
5	20	100
3π	7	21π
4	7e	28e

You can see the thing in common for the ones that do make 0, is that at least one of the numbers a and b is 0. From the second group you can see that when neither of them is 0, they won't multiply to make 0.

This makes total sense. If we multiply 0 by something, we are taking 0 of that thing, which will make zero. If we multiply a number that is not 0 by something, we will be getting a multiple of something (or part of something if it is between -1 and 1) which is definitely something, unless that is a 0 part. If we take no part of something, we have nothing.

To summarise, if the multiple of two numbers is 0, then at least one of those numbers must be 0.

Because our two brackets are simply numbers that multiply to make zero, they are like a and b in the above explanation. When two brackets multiply to make 0, one of those brackets must be 0.

So with $(x + 3)(x + 2) = 0$

We know that make 0 (on the RHS) either

$$x + 3 = 0 \text{ or } x + 2 = 0$$

And each of these are simple linear equations that we already know how to solve.

Equation	Do
$x+3=0$	(-3)
$x=-3$	

Equation	Do
$x+2=0$	(-2)

So the solutions of the equation $x^2+5x+6=0$ and the equation $(x+3)(x+2)=0$ are $x=-3$ and $x=-2$

After all that clever calculation, let's just check that the solutions do make the LHS = 0.

When $x=-3$
$x^2+5x+6=0$
Gives $9-15+6=0$
Which is true!

When $x=-2$
$x^2+5x+6=0$
Gives $4-10+6=0$
Which is true!

Hey presto we have just solved our first quadratic equation.

Let's try one more...

$x^2-3x-10=0$
First we factorise the LHS to give
$(x-5)(x+2)=0$
Because for two numbers multiplying to make 0 one of them must be zero, one of the linear bracket factors on the left hand side must be 0 so either...

$x-5=0$ or $x+2=0$

And each of these are simple linear equations that we already know how to solve.

Equation	Do
$x-5=0$	(+5)
$x=5$	

Equation	Do
$x+2=0$	(-2)

So the solutions of the equation $x^2-3x-10=0$ and $(x-5)(x+2)=0$ are the same, they are $x=5$ and $x=-2$. Check them to see if they make our original equation true.

The key steps to solving a quadratic are.

Step 0) Rearrange to see that your LHS is a quadratic and your RHS is 0
Step 1) Factorise the LHS
Step 2) Set each linear bracket factor equal to 0
Step 3) Solve these two linear equations

12) Solving Quadratic Equations when a≠1

For factorising quadratics, it is made significantly harder when your quadratic coefficient, your multiple of x^2 is anything other than one. For solving equations, it is not more difficult, as long as you have become fluent in factorising these more complex quadratics.

Let's try and solve $2x^2+11x+5=0$

Factorising the LHS we get
$(2x+1)(x+5)=0$

We now know that because make 0 (on the RHS) either$(2x+1)$ & $(x+5)$ multiply to make 0, one of them must be 0 so either...

$2x+1=0$ or $x+5=0$

Let's solve them

Equation	Do
$2x+1=0$	(-1)
$2x=-1$	(÷2)
$x=-\frac{1}{2}$	

Equation	Do
$x+5=0$	(-5)

So the solutions of $2x^2+11x+5=0$ and $(2x+1)(x+5)=0$ are the same, they are $x=-\frac{1}{2}$ and $x=-5$

Step 13) Solving Quadratics by Completing the Square

In the ladder on brackets you learned to put quadratics in a form called a completed square. In this form you get a single bracket squared, plus (positive or negative) a constant.

Because we don't have two different brackets (as in the factorised examples above) we can use this completed square form to solve equations by rearranging (just like quadratics that have a quadratic but not a linear term.

Equation	Do	Notes
$x^2+6x-7=0$	Complete the square	Use $\frac{b}{2}=\frac{6}{2}=3$ So $(x+3)^2-16$ $\equiv x^2+6x-7$
$(x+3)^2-16=0$	(+16)	
$(x+3)^2=16$	($\sqrt{\ }$)	
$x+3=\pm 4$	Solve both	So $x+3=4$ or $x+3=-4$
$x=1$ or $x=-7$		

Equation	Do	Notes
$2x^2+4x-3=0$	Complete the square	$2(x+1)^2-5 \equiv 2x^2+4x-3$
$2(x+1)^2-5=0$	(+5)	
$2(x+1)^2=5$	(÷2)	
$(x+1)^2=\frac{5}{2}$	($\sqrt{\ }$)	
$x+1=\pm\sqrt{\frac{5}{2}}$	Solve both	So $x+1=\sqrt{\frac{5}{2}}$ or $x+1=-\sqrt{\frac{5}{2}}$
$x=\sqrt{\frac{5}{2}}-1$ or $x=-\sqrt{\frac{5}{2}}-1$	Or in decimals (to 3dp)	So $x+1=1.581$ or $x+1=-1.581$
$x=0.581$ or $x=-2.581$		

Step 14) Solving Quadratics with the Quad Equations Formula

Not all quadratics can be factorised, but there is a lovely formula that lets us solve many of the quadratics that have solutions. Remember that the power of x tells us the maximum number of solutions to an equation, but there may be less or even no solutions. So a quadratic, can have 0, 1 or 2 solutions, but not 3 or more.

In the last step we saw that the solutions of $x^2+6x-7=0$ are $x=1$ and $x=-7$

And the solutions of

$2x^2+4x-3=0$ are $x=\sqrt{\frac{5}{2}}-1$ or $x=-\sqrt{\frac{5}{2}}-1$

We'll now use the formula to find these same solutions. After that we'll show you the wonderful bit of mathematics that explains what the formula means and where it comes from… it is brilliant! Here's the formula…

The solutions of $ax^2+bx+c=0$ are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So for our first example $x^2+6x-7=0$
Then $a=1, b=6, c=-7$
Substituting into the formula

$$x=\frac{-6\pm\sqrt{6^2-4\times1\times(-7)}}{2\times1}$$

$$x=\frac{-6\pm\sqrt{36--28}}{2}$$

$$x=\frac{-6\pm\sqrt{64}}{2}$$

$$x=\frac{-6\pm8}{2}$$

So $x=1$ and $x=-7$ as required.

And for $2x^2+4x-3=0$

Then $a=2, b=4, c=-3$
Substituting into the formula

$$x=\frac{-4\pm\sqrt{4^2-4\times2\times(-3)}}{2\times2}$$

$$x=\frac{-4\pm\sqrt{16--24}}{4}$$

$$x=\frac{-4\pm\sqrt{40}}{4}$$

$$x=\frac{-4}{4}\pm\frac{\sqrt{40}}{4}$$

$$x=\frac{-4}{4}\pm\frac{\sqrt{40}}{\sqrt{16}}$$

$$x=\frac{-4}{4}\pm\sqrt{\frac{40}{16}}$$

$$x=-1\pm\sqrt{\frac{5}{2}}$$

As required!

And here's where this magical formula comes from. You need to be sharp on completing the square, and on playing with surds!

The Quadratic Equations Formula to Solve $ax^2 + bx + c = 0$

Quadratic Equations Formula

Solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The y intercept of $y = ax^2 + bx + c$ are when x = 0, this gives y-intercept (0,c).

The x intercepts, known as roots, are found when y = 0, and are the solutions of $ax^2 + bx + c = 0$.

Equation	Do	Notes
$ax^2 + bx + c = 0$	**(÷a)**	
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	**complete the square**	$(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$
$(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$	$\left(+\frac{b^2}{4a^2} - \frac{c}{a}\right)$	
$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$	$\sqrt{}$	
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$	$(-\frac{b}{2a})$	**Note, as $\sqrt{}$ give us two possible, a +ve and a –ve solution, there are two solutions here represented by the $\pm$ symbol**
$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$	**Simplify RHS**	**We now have a usable formula, but the RHS needs to be simplified to be in the nice form we are used to seeing it.** **(See further notes below)**
$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$		Collect the fraction terms with common denominators
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$		

Notes on simplifying the surd.

Equation	Do	Notes
$\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$	**Create a common denominator**	$\frac{c}{a} = 1\frac{c}{a} = \frac{4a}{4a}\frac{c}{a} = \frac{4ac}{4a^2}$
$\sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}$	**And add the fractions inside the $\sqrt{}$**	
$\sqrt{\frac{b^2 - 4ac}{4a^2}}$	**Use surds to free top and bottom**	**In general** $\sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$
$\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$	**Simplify Bottom**	$\sqrt{4a^2} = 2a$ **as** $(2a)^2 = 2a \times 2a = 4a^2$
$\frac{\sqrt{b^2 - 4ac}}{2a}$	**As required!**	

Notes on completing the square...

Equation	Do	Notes
First guess is with $\frac{b}{2a}$ as this is half of the linear coefficient		
$(x + \frac{b}{2a})^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$	$(-\frac{b^2}{4a^2})$	**The first guess gives the correct quadratic and linear terms, but the constant needs adjusting**
$(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x$	$(+\frac{c}{a})$	**Now we have constant 0, so add the constant we want $(\frac{c}{a})$**
$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$		**Hey presto!**