w) Understanding Sequences

1 2 3 4 5 6

Step 1) Linear Sequences (Terms & Common Difference)

A sequence is a set of numbers in a particular order, where getting from one number to the next follows a pattern or rule. A linear sequence is one where the numbers go up or down by the same amount each time.

Here is an example of a linear sequence

3, 5, 7, 9, 11…

This is a set of numbers in order, and is not the same as 5, 7, 3, 8, 11… as the order has changed here.

Each of the numbers in a sequence is called a term. A sequence is made up of separate terms that come in a certain order just like the school year; you couldn't put the summer term before the winter term in a given academic year. The order of the terms always starts with the 1st term, then you have the 2nd term, and after that the 3rd term, followed by the 4th term, next the 5th term, and so on.

Going back to our example sequence we can label each term according to the order of the terms.

3,	5,	7,	9,	11…
1st Term	2nd Term	3rd Term	4th Term	5th Term

As we said the terms in a linear sequence go up or down by the same amount each time. We call the difference between two consecutive terms (that means two terms that are next to each other in the order of terms) the common difference. Common means the same, or in common. the same way we use the word in common factors, or with the lowest common multiple. We'll look at our example sequence and see what the difference is between each term.

3, 5, 7, 9, 11…

Difference is 5 – 3 = +2 | Diff. is 5 – 3 = +2 | Diff. is 5 – 3 = +2 | Diff. is 5 – 3 = +2

So these terms go up by 2 each time, we add two between each consecutive term. This means the difference is always the same, in other words the all consecutive terms have the same difference in common, so we call this a common difference of +2.

The … at the end of the sequence let's us know that the same pattern continues for ever. We can keep adding 2, and then add 2 some more… 3, 5, 7, 8, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29… 29 is the 15th term. We often just write down the first 5 terms as this is enough to show that the pattern is repeating without having to write down too many terms.

We can also have a negative common difference where we are subtracting a common number between each consecutive term.

17, 14, 11, 8, 5…

Difference is 14 – 17 = -3 | Diff. is 11 – 14 = -3 | Diff. is 8 – 11 = -3 | Diff. is 5 – 8 = -3

So here the difference between each term is -3, we are going down by the same amount, 3, between each consecutive term. So this is also a linear sequence.

You can create a linear sequence as long as you know the common difference and a term (usually the first term).

Write down the sequence with common difference 5, and first term 1.

+5 | +5 | +5 | +5

1, 6, 11, 16, 21…

1st Term is 1

2nd Term is 1st Term +5, 1 + 5 = 6

3rd Term is 2nd Term +5, 6 + 5 = 11

4th Term is 3rd Term +5, 11 + 5 = 16

5th Term is 4th Term +5, 16 + 5 = 21

So we can write "sequence with common difference +5, and first term 1" or we can write "1, 6, 11, 16, 21…" and these are a worded and a numerical way of describing exactly the same sequence.

We can even write a sequence down if we are given the common difference and a different term. Write down the sequence with common difference +10, and 3rd term 70. To find the 4th and 5th terms (and beyond) we just keep adding 10. But to get back from the 3rd, back to the 2nd term, we need to subtract 10. This is because to get from the 2nd term to the 3rd term we would +10, so to get from the 3rd to the 2nd, we do the opposite of +10, which is -10. To find the 1st term from the 2nd, we -10.

-10 | -10 | +10 | +10

50, 60, 70, 80, 90…

1st Term | 2nd Term | 3rd Term | 4th Term | 5th Term

Step 2) Sequences from Expressions

Amazingly, a linear expression like $3x - 1$, $2n + 3$, or $7 - 6y$ can also be used to express a sequence. So we have the worded and numerical ways from step 1, but we also have an algebraic way, writing our sequence as an expression.

The simple trick we use, is that the term number is used to replace the variable. So to find the value of a term, we substitute the term number into the expression.

Let's turn the expression $2n + 1$ into a numerical sequence.

Term Order	1st	2nd	3rd	4th	5th
Term Number	1	2	3	4	5
Calc.	2x1 + 1	2x2 + 1	2x3 + 1	2x4 + 1	2x5 + 1
Term	3	5	7	9	11…

Oh look, it is the same sequence we had from the first step: 3, 5, 7, 9, 11…

So this sequence can be described in 3 different ways.

1) In words: First term 3, common difference +2

2) In Numbers: 3, 5, 7, 9, 11…

3) In an Expression: $2n + 1$

We can do this in the same way for sequences written as expressions with a negative linear coefficient.

Let's work with expression $20 - 3n$

Term Order	1st	2nd	3rd	4th	5th
Term Number	1	2	3	4	5
Calc.	20 – 3x1	20 – 3x2	20 – 3x3	20 – 3x4	20 – 3x5
Term	17	14	11	8	5…

Wow! It is the negative sequence we had from the previous step: 17, 14, 11, 8, 5…

This sequence can also be explained in 3 different ways.

1) In words: First term 17, common difference -3

2) In Numbers: 17, 14, 11, 8, 5…

3) In an Expression: $20 - 3n$

We don't even need to work all the terms out in order. Let's find the 100th and the 37th term of these two sequences.

100th term has term number 100, so we substitute $n = 100$.

So for $2n + 1$
We get 2x100 + 1 = 201
So the 100th term of $2n + 1$ is 201

For $20-3n$
We get 20 – 3x100 = -280
So the 100th term of $20-3n$ is -280

And for the 37th term $n=37$ is the value to sub in to the expressions.

So for $2n+1$
We get 2x37 + 1 = 75
So the 37th term of $2n+1$ is 75

For $20-3n$
We get 20 – 3x37 = -91
So the 37th term of $20-3n$ is -91

Step 3) Sequences from Contexts

Sequences occur all over the place in nature: the patterns of petals on flowers; the hexagonal shaped growth of honeycomb; populations of animals in the wilderness. There are many biologists who study these patterns who have to be experts in this area of mathematics to study and explain the patterns nature the way the do.

Let's look at a stack of chairs.

The height of the stack is not simply the height of one chair times the number of chairs, as the chairs partly fit inside one another.

Let's say the chairs height is 50cm, but they stack 10cm apart (oeverlapping by 40cm when stacked).

One chair alone will (obviously I hope) be 50cm high. But the second chair will add only 10cm to the total height because 40cm of chair overlaps with the first chair. And the third chair will also overlap with the existing stack and add only another 10cm to the stack.

Number of Chairs	Calculation	Stack Height
1	50	50
2	50+10	60
3	50+10+10	70
4	50+10+10+10	80
5	50+10+10+10+10	90

So this is a linear arithmetic sequence, as the same amount (10cm) is added to the height each time you add a chair to the stack.

The situation can be represented by sequence

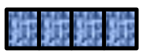
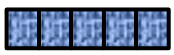
50, 60, 70, 80, 90…

Or expression 10n + 40

Staying with the theme of chairs, lets look at the number of people gathered around some square tables in a restaurant organised in a long line.

Around the first square table, which has 4 sides, you can fit 4 chairs, and hence 4 people.

If you added a second table without seperately you would then be able to seat 8 people. But when you put the two tables together to make one long table, you have to put two table edges together, so you lose two chairs compared to having the two tables seperately. You seat only six people. Now for every extra table you only add 2 seats, as you lose one off the end of the existing table, and only gain 3 places with the new table.

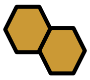
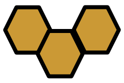
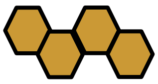
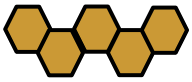
Number of Tables	Picture	Calculation	People
1		4	4
2		4+2	6
3		4+2+2	8
4		4+2+2+2	10
5		4+2+2+2+2	12

This sequence can be represented numerically (in numbers) as 4, 6, 8, 10, 12… or Algebraicly (using algebraic variables, or letters) as 2n + 2

Yum, Honeycombe!

Honeycomb is built of hexagons. The hexagon shape is very important. Every wall the bees build takes their energy and time. A hexagon makes the strongest shape with the least amount of material.

Let's look at a string of hexagons built horizontally. To build the first hexagon the bees need to build 6 sides. But for every hexagon after that they only need 5 more sides, as they add the shape on to one of the existing sides.

Hexagons	Picture	Pattern	Sides
1		6	6
2		6+5	11
3		6+5+5	16
4		6+5+5+5	21
5		6+5+5+5+5	26

This situation can bee (ha ha) represented by the sequence…

6, 11, 16, 21, 26…

It is interesting to note that with the tables problem you only get the number of sides around the edge of the shape as you are interested in how many people you can seat. But with the honeycomb problem, you are interested in how many walls there are, so the connecting sides also count.

Step 4) Expressions from Linear Sequences

Let's take a look at a number of linear sequences changing the multiple of n, but not the constant term, and see how this effects the common difference

2n + 1 = 3, 5, 7, 9, 11… has common difference 2

3n + 1 = 4, 7, 10, 13, 16… has common difference 3

4n + 1 = 5, 9, 13, 17, 21… has common difference 4

and

5n + 1 = 6, 11, 16, 21, 26… has common difference 5

It looks like the common difference is always the same as the multiple of n in the linear term. This is actually always true. The multiple of n in the linear term (of the expression) comes from the fact that we have added the common difference a number of times, and this is represented by the multiple of n term.

Let's look back again at our examples from the last step, and see how the addition of the common difference each time can be written as a multiple of n (the term number).

Number of Chairs	Calculation	As multiple	Stack Height
1	50	50 or 1x10 + 40	50
2	50+10	10 + 50 or 2x10 + 40	60
3	50+10+10	2x10 + 50 or 3x10 + 40	70
4	50+10+10+10	3x10 + 50 or 4x10 + 40	80
5	50+10+10+10+10	4x10 + 50 or 5x10 + 40	90

You can see that we can't just use the first term as the constant in our expression. We need 1 lot of our common difference and the constant if we are going to use n = 1 in our first term. Above you can see that the constant is 40, as the first term is 50 and the common difference is 10. One lot of this common difference, means the constant needs to allow another 40 to get to our first term.

So in general we can find an expression for an arithmetic sequence by using the common difference as the multiple of n in the linear term of the expression and then working out what our constant is from one of the terms.

Let's look at

5, 9, 13, 17, 21…

The common difference is +4 so will will look at the sequence represented by 4n + c. We just need to find out what the constant c is.

The first term, 5 = 4 x 1 + c
This means c must be 1

This can be done with any term

2nd Term: 9 = 4x2 + c,   so c = 1

3rd Term: 13 = 4x3 + c,   so c = 1

4th Term: 17 = 4x4 + c, so c = 1

5th Term: 21 = 4x5 + c, so c = 1

100th Term: 401 = 4x100 + x, so c = 1

Any term we use, we get the same constant (this must be true if the same expression is going to work for any term by substituting in the term number).

So 5, 9, 13, 17, 21… can be represented algebraically by expression 4n + 1.

Let's look at 3, 7, 11, 15, 19…

Again the common difference it +4 so we'll look at 4n +c in the first term.

5 = 4x1 + c

Now c = -1

So 3, 7, 11, 15, 19… is the same as 4n – 1

This same method works when the multiple of n (the linear coefficient) is negative.

20, 17, 14, 11, 8…

The common difference is -3, so we'll use -3n + c

1st Term: 20 = -3x1 + c, so c = 23

Hence 20, 17, 14, 11, 8… = -3n + 23

Another way to thin about turning a sequence, say 5, 7, 9, 11, 13… into an expression, is to first use the common difference to find the linear term, and then find the difference between this and the actual term. You can call this common difference times n stage a first try.

5, 7, 9, 11, 13… has common difference +2 so we will try 2n, and see what we need to + or – to get our actual sequence.

2n = 2, 4, 6, 8, 10…

+3 +3 +3 +3 +3

Our sequence = 5, 7, 8, 11, 13…

We need to add 3 to our first try of 2n,
so 5, 7, 8, 11, 13… is 2n + 3

This way of thinking about it, using the common difference to find first try, and then adjust it to get to your actual sequence will prove invaluable with quadratic and higher sequences.

Step 5) Expressions from Quadratic Sequences (b=0)

In the same way that if investigating patterns with whole numbers we start with 1, and then work with 2, then 3 and so on. With algebra we misght start with constants, then llinears, then quadratics, then cubbbics and so on. So the simplest type of quadratic expression only has a quadratic term. Let's investigate the common differences of quadratic terms.

$\underline{n^2}$

1, 4, 9, 16, 25

+3 +5 +7 +9

We might be tempted to give up at this point, because the differences aren't common. They aren't all the same but are actually changing. You may notice the the differences themselves form a linear sequence. Let's try looking at the second differences; that is the differences between differences of the terms of our actual sequence.

1, 4, 9, 16, 25

+3 +5 +7 +9

+2 +2 +2

So there is a common second difference for the sequence n^2, and that common second difference is +2.

Let's also look at $2n^2$, $3n^2$, $4n^2$, and $5n^2$

$\underline{2n^2}$

2, 8, 18, 32, 50

+6 +10 +14 +18

+4 +4 +4

So $2n^2$ has a common second difference of +4.

$\underline{3n^2}$

3, 12, 27, 48, 75

+9 +15 +21 +27

+6 +6 +6

$3n^2$ has a common second difference of +6.

$\underline{4n^2}$

4, 16, 36, 64, 100

+12 +20 +28 +36

+8 +8 +8

$4n^2$ has a common second difference of +8.

$4n^2$

5, 20, 45, 80, 125

+15 +25 +35 +45

+10 +10 +10

$5n^2$ has a common second difference of +10.

Let's put these common second differences into a table and look for a pattern.

Term	Co-efficient of n^2	Common Second Difference
n^2	1	2
$2n^2$	2	4
$3n^2$	3	6
$4n^2$	4	8
$5n^2$	5	10

We can see that the common second difference is always double the quadratic co-efficient. Let's use this to try and find an expression to represent quadratic sequence 5, 11, 21, 25, 53.

Firstly we will work out what the common second difference is.

5, 11, 21, 35, 53

+6 +10 +14 +18

+4 +4 +4

The common second difference is +4, and the common second difference was always double the quadratic co-efficient, so let's halve it.

Try $\frac{4}{2}n^2 = 2n^2$

$2n^2$ = 2,8,18,32,50...

common 2nd difference

is always halved

Which isn't our sequence ☹. Have we failed? No!

Let's compare our first try sequence of $2n^2$ in a table with our actual sequence and see what the difference is?

First try of $2n^2$ = 2, 8, 18, 32, 50...

+3 +3 +3 +3 +3

Actual sequence = 5, 11, 21, 35, 53

So the terms of our sequence are each 3 more than the terms of our first try $2n^2$

So 5,11,21,35,53...
Is represented by expression $2n^2 + 3$

6) Expressions from Quadratic Sequences (b≠0)

Express the n^{th} term of 4, 13, 26, 43, 64...

Firstly let's look at the first and second differences.

4, 13, 26, 43, 64

+9 +13 +17 +21

+4 +4 +4

There is a common second difference of +4, so we will make our first try $2n^2$

First try $2n^2$ = 2, 8, 18, 32, 50...

+2 +5 +8 +11 +14

Our sequence = 4, 13, 26, 43, 64...

The difference between our first try of $2n^2$ and our actual sequence of 4, 13, 26, 43, 64... is not just a constant. So our final expression wont be as simple as $2n^2 + c$, it must have a linear term too.

Let's look at the increasing differences between $2n^2$ and our actual quadratic sequence and treat them as a linear sequence.

This sequence of differences is

2, 5, 8, 11, 14

+3 +3 +3 +3

And this linear sequence is $3n + c$

Using the first term 2 = 3n + c, so c = -1

So 2, 5, 8, 11, 14.... is $3n - 1$

Hence the linear sequence we need to add to our first try of $2n^2$ is $3n - 1$

So our actual quadratic sequence 4, 13, 26, 43, 64... can be found by adding the terms of:

First try $2n^2$ found using the common second difference of our actual sequence

and

$3n - 1$ found using the sequence of the differences between our first try and our actual sequence

So 4, 13, 26, 43, 64...

is represented by quadratic expression

$$2n^2 + 3n - 1$$

Now investigate some cubic sequences.